Numerical Algebraic Geometry via Numerical Polynomial Algebra

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SIAM Conference on Applied Algebraic Geometry – AG11
North Carolina State University
Raleigh North Carolina
October 2011

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H-Basis

- **Möller and Sauer**

  At a very early time, when even the notion of ideals was not commonly accepted, Macaulay introduced H-bases. *These special bases of polynomial ideals are also helpful in various branches of numerical analysis.*

- **Macaulay**

  The distinctive property of an H-basis $(F_1, F_2, \ldots, F_k)$ of M is than any member F of M can be put in the form $A_1F_1 + A_2F_2 + \cdots + A_kF_k$ where A_iF_i $(i = 1, 2, \ldots, k)$ is not of greater degree than F. Every module [ideal] has an H-basis, which may necessarily consist of more members than would suffice for a basis in general

Note: A homogeneous basis for a homogeneous ideal is an H-basis, original motivation for name "H-basis"

Macaulay and Sylvester Arrays

Let $f = x - y$, $g = z + x^2 - y^2$ in $\mathbb{C}[x, y, z]$, $F = [f, g]$

Macaulay Array [DLZ] of F of order 2 at $(0, 0, 0)$

	1	x	y	z	x^2	xy	xz	y^2	yz	z^2
f	0	1	-1	0	0	0	0	0	0	0
g	0	0	0	1	1	0	0	-1	0	0
xf	0	0	0	1	-1	0	0	0	0	0
yf	0	0	0	0	0	1	0	-1	0	0
zf	0	0	0	0	0	0	1	0	-1	0
xg	0	0	0	0	0	0	1	0	0	0
yg	0	0	0	0	0	0	0	0	1	0
zg	0	0	0	0	0	0	0	0	0	1

Note that rows xg, yg, zg are truncated.

Macaulay and Sylvester Arrays (continued)

Sylvester Array of F of order 2, $\mathbf{S}(F, 2)$

	1	x	y	z	x^2	xy	xz	y^2	yz	z^2
f	0	1	−1	0	0	0	0	0	0	0
g	0	0	0	1	1	0	0	−1	0	0
xf	0	0	0	1	−1	0	0	0	0	0
yf	0	0	0	0	0	1	0	−1	0	0
zf	0	0	0	0	0	0	1	0	−1	0

Macaulay and Sylvester Arrays (continued)

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xf	0	0	0	1	−1	0	0	0	0	0
yf	0	0	0	0	0	1	0	−1	0	0
zf	0	0	0	0	0	0	1	0	−1	0

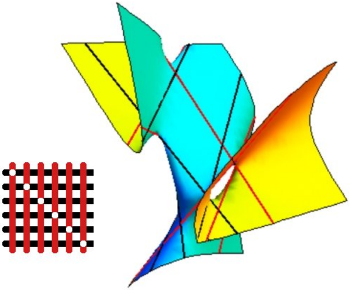
Sylvester Array of ideal $\langle f, g\rangle$ of order 2, $\mathbf{S}(\mathcal{I}, 2)$

	1	x	y	z	x^2	xy	xz	y^2	yz	z^2
$\mathbf{x} - \mathbf{y}$	0	1	−1	0	0	0	0	0	0	0
$\mathbf{z}$	0	0	0	1	0	0	0	0	0	0
$x(x-y)$	0	0	0	0	1	−1	0	0	0	0
$y(x-y)$	0	0	0	0	0	1	0	1	0	0
xz	0	0	0	0	0	0	1	0	0	0
yz	0	0	0	0	0	0	0	0	1	0
z^2	0	0	0	0	0	0	0	0	0	1

Intersections of Ideals via Sylvester Arrays

Lemma: *The rowspace of the Sylvester Array of the ideal $I \cap J$ is the intersection of the rowspaces of the Sylvester Arrays of I, J.*

This is the main tool used in my Sommese Conference paper [D1] where I construct ideals of configurations of lines in a non-singular cubic surface in $\mathbb{P}^3$. Ideal generators are given by polynomials with floating point coeffficients.



This picture shows Schläfli's double 6 via Hilbert's construction.

Global and Local dual functionals

Let $\mathcal{I}$ be an ideal of $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \ldots, x_s]$, the local ring at $\hat{\mathbf{x}} = 0$ is $\mathbb{C}[[x_1, \ldots, x_s]]/\, \mathbb{C}\, [[x_1, \ldots, x_s]]\mathcal{I}$. For $\mathbf{j} = [j_1, \ldots, j_s], \mathbf{x}^{\mathbf{j}} = x_1^{j_1} \ldots x_s^{j_s}$.

A *Global dual functional*, in my terminology, is a $\mathbb{C}$-linear map

$$\mathbb{C}[\mathbf{x}]/\mathcal{I} \longrightarrow \mathbb{C}$$

A typical such functional is given by an **infinite** sum

$$\sum_{\mathbf{j}} \alpha_{\mathbf{j}} \mathbf{X}^{\mathbf{j}} \text{ where } \mathbf{X}^{\mathbf{j}}(\mathbf{x}^{\mathbf{k}}) = \begin{cases} 1 & \text{if} \quad \mathbf{j} = \mathbf{k}, \\ 0 & \text{if} \quad \mathbf{j} \neq \mathbf{k}. \end{cases}$$

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A *Local dual functional* is a $\mathbb{C}$-linear map

$$\mathbb{C}[[\mathbf{x}]]/\mathbb{C}[[\mathbf{x}]]\mathcal{I}\Big|_{\hat{\mathbf{x}}} \longrightarrow \mathbb{C}$$

A typical such functional is given by a **finite** sum

$$\sum_{|\mathbf{j}|<n} \beta_{\mathbf{k}} \partial_{\mathbf{x}^{\mathbf{j}}}[\hat{\mathbf{x}}], \text{ where } \partial_{\mathbf{x}^{\mathbf{j}}}[\hat{\mathbf{x}}] \equiv \frac{1}{j_1! \cdots j_s!} \left.\frac{\partial^{j_1+\cdots+j_s}}{\partial x_1^{j_1} \cdots \partial x_s^{j_s}}\right|_{\hat{\mathbf{x}}}$$

Global and Local dual functionals spaces as arrays

Local duals can be put in Sylvester type arrays, global in Macaulay type. We view the dual functionals as columns.
Consider the ideal $\langle f \rangle \subseteq \mathbb{C}[x, y]$ given by
$f = x + 2y + x^2 + 3xy + y^2$. The local duals are at point $(0, 0)$, indices on right.

Local duals order 2

1	0	0	∂_1
0	-2	1	∂_x
0	1	0	∂_y
0	0	4	∂_{x^2}
0	0	-2	∂_{xy}
0	0	1	∂_{y^2}

Global duals order 2

1	0	0	0	0	1
0	-2	1	1	0	X
0	1	0	0	0	Y
0	0	4	-4	-3	X^2
0	0	-2	1	1	XY
0	0	1	0	0	Y^2

Note the last two columns of the global duals are truncated.

Local and Global Duality

The notion of global dual functional $\mathbb{C}[\mathbf{x}]/\mathcal{I} \to \mathbb{C}$ implies that dual functionals kill the ideal $\mathcal{I}$ (and similarly for local duals).
From this viewpoint say matrices A, B are *dual* if $AB = 0$ with rowspace A the left nullspace of B and column space B the nullspace of A.

The Sylvester array of local duals is dual to the Macaulay matrix while the the Macaulay array of Global duals is dual to the Sylvester array of the ideal.

Local to Global

For $\mathbf{i} = [i_1, \ldots, i_s], \mathbf{j} = [j_1, \ldots, j_s]$, $\mathbf{i} \geq \mathbf{j}$ means $i_\alpha \geq j_\alpha$ for all $1 \leq \alpha \leq s$. Then as functionals on $\mathbb{C}[\mathbf{x}]$

$$\partial_{\mathbf{x}^\mathbf{j}}[\hat{\mathbf{x}}] = \sum_{\mathbf{i} \geq \mathbf{j}} \binom{i_1}{j_1} \hat{x}_1^{i_1 - j_1} \cdots \binom{i_s}{j_s} \hat{x}_s^{i_s - j_s} \mathbf{X}^\mathbf{i}$$

where $\hat{\mathbf{x}} = (\hat{x}_1, \ldots, \hat{x}_s)$. The left hand side is a local functional and the right a global functional. From a matrix point of view we have for fixed n

$$\begin{bmatrix} \text{Macaulay Matrix} \\ \text{of order } n \\ \text{global duals from } \hat{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} \text{Change of Center} \\ \text{matrix} \\ \text{of order } n \end{bmatrix} \begin{bmatrix} \text{Sylvester Matrix} \\ \text{of order } n \\ \text{local duals at } \hat{\mathbf{x}} \end{bmatrix}$$

Local to Global

Change of Center Matrix

For example if $s = 2$ and $\hat{\mathbf{x}} = (1, 2)$ then the *Change of center matrix* is

$$\gamma_{\hat{\mathbf{x}}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 3 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 4 & 1 & 2 & 2 & 0 & 0 & 1 & 0 & 0 \\ 4 & 4 & 4 & 0 & 4 & 1 & 0 & 0 & 1 & 0 \\ 8 & 0 & 12 & 0 & 0 & 6 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Local to Global, Main Theorem

Given an ideal $\mathcal{I}$ of $\mathbb{C}[x_1, \ldots, x_s]$, $n > 0$ and points $\hat{\mathbf{p}}_i, \; i = 1, \ldots, k$ of $V(\mathcal{I})$ concatenate the Macaulay matrices of order n global duals. Write $\mathcal{D}_n(\{\hat{\mathbf{p}}_1, \ldots, \hat{\mathbf{p}}_k\})$ for this matrix.

Main Theorem, matrix form: *For given $n > 0$ there exist finitely many points, $\hat{\mathbf{p}}_1, \ldots, \hat{\mathbf{p}}_k$, of $V(\mathcal{I})$ so that the Sylvester matrix of the ideal $\mathcal{I}$ of order n is the left nullspace of $\mathcal{D}_n(\{\hat{\mathbf{p}}_1, \ldots, \hat{\mathbf{p}}_k\})$.*

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Corollary *An H-Basis for $\mathcal{I}$ can be obtained from finitely many local duals at finitely many points of $V(\mathcal{I})$.*

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Corollary *An H-Basis for $\mathcal{I}$ can be obtained from finitely many local duals at finitely many points of $V(\mathcal{I})$.*

It remains an open question as to how many and what points are needed. It is clear that it is necessary to have at least one point from each component of $V(\mathcal{I})$. In principle, for large n, that may be enough. In practice more points may be needed and the number may be dependent on implementation issues as well as algebraic-geometric factors.

The global Hilbert function

The global Hilbert function is

$$\mathrm{GHF}(n) = \binom{n+s}{s} - \mathrm{rank}\ \mathbf{S}(\mathcal{I}, n),\ \ n > 0$$

But if $\hat{\mathbf{p}}_1, \ldots, \hat{\mathbf{p}}_k$ satisfy the Main Theorem then

$$\mathrm{GHF}(n) = \mathrm{rank}\ \mathcal{D}_n(\{\hat{\mathbf{p}}_1, \ldots, \hat{\mathbf{p}}_k\}),\ \ n > 0$$

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For large n the values of the Hilbert function agree with a integer valued polynomial known as the Hilbert Polynomial. This can often be calculated independently. In particular the leading term $c_d t^d$ of this polynomial can often be calculated by standard Numerical Algebraic geometry software. In general one wants to pick points and tolerances minimizing the global Hilbert function retaining the correct leading term of the Hilbert Polynomial.

Numerical Issues

- **Implementation:** Sparse matrix methods are used with MATHEMATICA 8. Generally the built in algorithms that accept a tolerance option are used, eg. `MatrixRank`, `NullSpace`. SVD is used for finding rowspaces. Typical timing to find a Macaulay matrix of degree 8 for a system of 4 equations in 4 variables (a 1320×495 sparse matrix) is under 10 seconds, finding the null space takes about 1.5 seconds.
- **Non-deterministic** It is hard to predict exactly how many and which points and tolerance is needed to satisfy the Main Theorem. For example the Cyclic-4 system has a curve of degree 4 as its solution set but small pertubations generally give a zero-dimensional system. So this system is numerically unstable. Experiments in [D2] show one may need 4 or 5 random points and a tolerance between $10^{-8} and 10^{-12}$ to get a 90% chance of correct Hilbert Function depending on the Mathematica version used.

Algorithms

The algorithms for finding $\mathcal{D}_n(\{\hat{\mathbf{p}}_1, \ldots, \hat{\mathbf{p}}_k\})$ (see also work of Mourrain, Li and Zhi, Zeng) and $\mathbf{S}(\mathcal{I}, n)$ are straight forward.

Two algorithms have been used for extracting H-bases.
MBasis1

- Calculate $\mathcal{D}_N(\{\hat{\mathbf{p}}_1, \ldots, \hat{\mathbf{p}}_k\})$ for N large enough that $\mathbf{S}(\mathcal{I}, N)$ contains an H-Basis. For $n \leq N$ $\mathcal{D}_n(\{\hat{\mathbf{p}}_1, \ldots, \hat{\mathbf{p}}_k\})$ is truncation of $\mathcal{D}_N(\{\hat{\mathbf{p}}_1, \ldots, \hat{\mathbf{p}}_k\})$ and $\mathbf{S}(\mathcal{I}, n) = \text{left nullspace } \mathcal{D}_n(\{\hat{\mathbf{p}}_1, \ldots, \hat{\mathbf{p}}_k\})$.
- For $n = 1, 2, \ldots$ calculate $\mathbf{S}(\mathcal{I}, n)$ until $\mathbf{S}(\mathcal{I}, n_0)$ is non-empty. Interpret entries of $\mathbf{S}(\mathcal{I}, n_0)$ as polynomials and set $\mathcal{B}_{n_0}$ to be this list of polynomials.
- For $n_0 < n \leq N$ note $\mathbf{S}(\mathcal{B}_{n-1}, n) \subseteq \mathbf{S}(\mathcal{I}, n)$. If this inequality is an equality set $\mathcal{B}_n = \mathcal{B}_{n-1}$. Otherwise there are rows of $\mathbf{S}(\mathcal{I}, n)$ independent of $\mathbf{S}(\mathcal{B}_{n-1}, n)$ and add corresponding polynomials to $\mathcal{B}_{n-1}$ to obtain $\mathcal{B}_n$.
- **Output:** $\mathcal{B}_N$

Algorithms

MBasis2

Example: $\mathcal{I} = \langle 1 + x + y + xy^2, 1 - y^2 \rangle$

Calculate $\mathbf{S}(\mathcal{I}, 3)$, put this in reverse RREF form:
(monomial order $1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3$)

$$\left[\begin{array}{cccccccccc}
\mathbf{1} & \mathbf{2} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]$$

The first row is $\mathbf{S}(\mathcal{I}, 1)$, the first 4 rows form $\mathbf{S}(\mathcal{I}, 2)$ and the entire matrix is $\mathbf{S}(\mathcal{I}, 3)$. As in algorithm **MBasis1** the minimal H-basis is $1 + 2x + y, \; x + x^2$.

Here numerical issues have been ignored, see [D2].

Example from Shen and Yaun [SY]

Consider the parametric space curve $P(t) = [p_1(t), p_2(t), p_3(t)]$

$$p_1(t) = t^2(t-2), p_2(t) = (t-1)^2(t+1), p_3(t) = t(t-1)(t-2)$$

Let $f = \mathrm{RES}(x - p_1, y - p_2, t)$, $g = \mathrm{RES}(x - p_1, z - p_3, t)$, $h = \mathrm{RES}(y - p_2, z - p_3, t)$ where RES is the classical resultant wrt t. Then the curve $P(t)$ is contained in $V(F) = V(\langle f, g, h\rangle)$

$$f = -3 - 7x - 5x^2 - x^3 + 7y + 9xy + 3x^2y - 5y^2 - 3xy^2 + y^3$$

$$g = -x^2 - x^3 + 2xz + 3x^2z - 3xz^2 + z^3$$

$$h = -3y + 4y^2 - y^3 - 2yz + 3y^2z + 6z^2 - 3yz^2 + z^3$$

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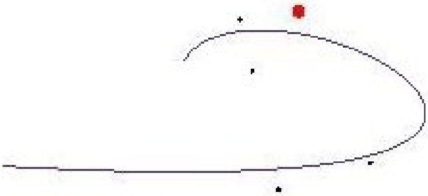
$V(\langle f, g, h\rangle)$ has 5 extra points, one of them a double point



Example continued

To find the system which does implicitly define the curve we pick 3 random points on the curve which is easy: pick random values of t.

$$P1 = \{-1.181304680, 0.1908788777, -0.2647021088\}$$
$$P2 = \{-2.867881911, 0.0750806185, -5.7917099074\}$$
$$P3 = \{0.541961237, 3.9180659782, 0.2863825648\}$$

Using Mathematica 8:

Timing[G8=GDiff[F,{P1,P2,P3},8,X, ep]; MBasis[G8,3,X,ep]]
Final Tolerance = 1.`*^-12
Degrees {2,2,2}
Affine Hilbert Function {1,4,7,10}

$$\{7.668491,$$
$$\{3. + 6.x + 3.x^2 - 4.y - 3.xy + 1.y^2 - 1.z - 1.xz,$$
$$-1.x - 1.x^2 - 1.z + 1.yz,$$
$$3.x + 3.x^2 - 1.xy - 3.xz + 1.z^2\}\}$$

Example Concluded

In a similar way we calculate the system defining the 6 (counting multiplicity) extraneous points.

$$-0.511796 - 0.713922x + 0.200749x^2 + 0.597126y + xy - 0.0860177z$$
$$0.342988 + 0.591234x + 0.237358x^2 + 0.010888y + 0.888927z + xz$$
$$0.543137 + 0.579456x + 0.297696x^2 - 1.26138y + 1.y^2 + 0.385248z$$
$$-0.279985 - 0.101178x - 0.0939662x^2 + 0.272773y - 0.950368z + yz$$
$$-0.0710982 - 0.0425179x + 0.0579917x^2 - 0.0294109y + 0.143796z + z^2$$

Intersecting the ideals of these systems we get a H-Basis for F consisting of the 3 original equations plus the real polynomial

$$\begin{aligned}0.560871 + 5.20633x + 12.6412x^2 + 10.9957x^3 + 3x^4 - 0.747828y - 4.23234xy \\ - 6.81499x^2y - 3x^3y + 0.186957y^2 + 0.856481xy^2 + x^2y^2 \\ + 0.591564z - 0.0904207xz - 2.59311x^2z - x^3z - 0.778521yz \\ - 1.68964xyz + 0.245542z^2 + 0.245542xz^2\end{aligned}$$

Note that an H-basis calculated for F by the Gröbner Basis method consists of 6 or more integer polynomials with huge coefficients.

References

B. H. Dayton, T-Y Li, Z. Zeng *Multiple Zeros of nonlinear systems* Math. Comp. 80 (2011), no. 276, pp. 2143-2168.

B. H. Dayton, *Ideals of Numeric Realizations of Configurations of Lines*, Contemp. Math. 490, (2009). pp. 181–198.

B. H. Dayton, *Numerical Calculation of H-bases for Positive Dimensional Varieties*, Proceedings of SNC2011, ACM Press, 2011.

F. S. Macaulay *The Algebraic Theory of Modular Systems*, Cambridge University Press, 1916.

H. M. Möller and T. Sauer, *H-bases for polynomial interpolation and system solving*, Adv. in Comp. Math. 12 (2000), pp. 335-362.

L. Shen, C. Yaun, *Implicitization using Univariate Resultants*, J Sys Sci Complex (2010) 23, pp. 804-814.