The Functorial Property of the Global Dual

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The Global Dual

Given a $\mathbb{C}$-algebra $\mathcal{A}$ the global dual $\mathcal{G}(\mathcal{A})$ is the $\mathbb{C}$-vector space of linear maps $d : \mathcal{A} \to \mathbb{C}$.

Example: Let $\mathcal{A} = \mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \ldots, x_s]$. Then $\mathcal{G}(\mathcal{A})$ can be identified with the $\mathbb{C}$-vector space $\mathbb{C}[[X_1, \cdots, X_s]]$ of formal power series where, for $\mathbf{X}^{\mathbf{k}} = X_1^{k_1} \cdots X_s^{k_s}$ and $\mathbf{x}^{\mathbf{j}} = x_1^{j_1} \ldots x_s^{j_s}$,

$$\mathbf{X}^{\mathbf{k}}(\mathbf{x}^{\mathbf{j}}) = \begin{cases} 1 & \text{if} \quad \mathbf{j} = \mathbf{k}, \\ 0 & \text{if} \quad \mathbf{j} \neq \mathbf{k}. \end{cases}$$

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More generally if $\mathcal{A} = \mathbb{C}[\mathbf{x}]/\mathcal{I}$ then $\mathcal{G}(\mathcal{A})$ is the subspace of $\mathcal{G}(\mathbb{C}[\mathbf{x}])$ given by $\mathcal{G}(\mathcal{A}) = \{d \mid d(f) = 0 \text{ for all } f \in \mathcal{I}\}$

The Functorial Property

Suppose $\mathcal{X} = V(\mathcal{I}) \subseteq \mathbb{A}^s$ and $\mathcal{Y} = V(\mathcal{J}) \subseteq \mathbb{A}^r$ and

$$\phi : \mathbb{A}^s \longrightarrow \mathbb{A}^r$$

is an affine map $\phi = [f_1, \ldots, f_r]$ where $f_i : \mathbb{C}[x_1, \ldots, x_s] \to \mathbb{C}$ are polynomial functions such that $\phi(\mathcal{X}) \subseteq \mathcal{Y}$. This gives a ring map

$$\phi^* : \mathbb{C}[y_1, \ldots, y_r]/\mathcal{J} \longrightarrow \mathbb{C}[x_1, \ldots, x_s]/\mathcal{I}$$

given by $\phi^*(g) = g(f_1, \ldots, f_r)$. Then we get a linear map

$$\phi_* : \mathcal{G}(\mathbb{C}[\mathbf{x}]/\mathcal{I}) \longrightarrow \mathcal{G}(\mathbb{C}[\mathbf{y}]/\mathcal{J})$$

defined by $\phi_*(d) = d(\phi^*)$.

It is seen that we have a covariant functor from algebraic sets $(\mathcal{X}, \mathcal{I})$ and affine maps to $\mathbb{C}$ vector spaces and linear maps.

Macaulay and Sylvester Arrays of Polynomials and Ideals

Caution: The notation here is evolving an may not be fully consistent with earlier work by the author.

Let $f = x - y + 3y^3,\ g = z + 2x^2 - 3y^2$ in $\mathbb{C}[x, y, z], \quad F = [f, g, yg]$

The *Macaulay Array* [DLZ] of F of order 2 at the origin is

	1	x	y	z	x^2	xy	xz	y^2	yz	z^2
f	0	1	-1	0	0	0	0	0	0	0
g	0	0	0	1	2	0	0	-3	0	0
yg	0	0	0	0	0	0	0	0	1	0

Note that the rows for f, yg are truncated. If we also included rows for xf, xg, yf, zf, zg we would call this the *full Macaulay array of order 2*.

Macaulay and Sylvester Arrays Continued

- If only rows which correspond to polynomials of order n, i.e. not truncated, are included we call the array a *Sylvester Array of order n*. If all multiples of these rows by monomials which still have terms not exceeding total degree n are included we have the *full Sylvester Array of order n*.
- If we include additional rows so that every polynomial of total degree n or less in the ideal generated by the listed polynomials (eg. F in the previous page) then we would call this the *Sylvester Matrix of the ideal.* In the example of the previous page one would include the row corresponding to the polynomial $f + yg = x - y + z + 2x^2$.
- If the Sylvester Matrix of a list F is a Sylvester matrix of the ideal generated by the elements of F then F is an *H-basis*. This concept is due to Macaulay [Mac] and is useful for numerical work [MS].

Macaulay Arrays of Duals and Dual Principle

Because of the representation of duals $\Sigma X^{\mathbf{i}}$ for quotients of polynomial rings we can also define Macaulay arrays of duals. In this case we use a transposed Macaulay matrix, i.e. the row indices are the $X^{\mathbf{i}}$ and and the dual vectors correspond to columns. We will write $\mathcal{G}_n(\mathcal{A})$ for the dual Macaulay matrix of order n.

Dual Principle:

$$\begin{bmatrix} \text{Sylvester Matrix} \\ \text{of ideal } \mathcal{I} \end{bmatrix} \begin{bmatrix} \text{Macaulay} \\ \text{array of} \\ \text{global} \\ \text{duals of} \\ \mathbb{C}[\mathbf{x}]/\mathcal{I} \end{bmatrix} = 0$$

The columns of the Macaulay matrix span the nullspace of the Sylvester matrix and the rows of the Sylvester matrix span the left nullspace of the Macaulay matrix.

Computational Procedure

To construct dual Macaulay matrix of $\mathbb{C}[\mathbf{x}]/\mathcal{I}$

- Produce an H-basis of $\mathcal{I}$ by methods in [Mac, MS] or by global degree Gröbner Bases.
- Construct Sylvester matrix of the ideal of appropriate size n.
- Extract the nullspace as a column matrix $\mathcal{G}_n(\mathbb{C}[\mathbf{x}]/\mathcal{I})$.

OR

- Use the local-global method of [D1, D2].

To constuct $\mathcal{I}$ from $\mathcal{G}_n(\mathbb{C}[\mathbf{x}]/\mathcal{I})$

- Take left nullspace of $\mathcal{G}_n(\mathbb{C}[\mathbf{x}]/\mathcal{I})$ to get Sylvester Matrix.

OR

- Use algorithm MBasis from [D1, D2] to find minimal H-basis.

Implementation details such as choice of n will not be discussed in this lecture.

The transformation $\mathcal{G}_n(\phi)$

Assume that $\phi = [f_1, \ldots, f_r] : \mathbb{A}^s \to \mathbb{A}^r$ is a polynomial map which takes the origin to the origin. If necessary do a linear translation of variables or homogenize. Let $\mathcal{X} = V(\mathcal{I})$ be an algebraic set in $\mathbb{A}^s$ and set $\mathcal{J} = \phi^{*-1}(\mathcal{I})$ so that $\mathcal{Y} = \overline{\phi(\mathcal{X})}$.

Construct $\mathcal{G}_n(\phi)$ as follows:

- For each monomial $\mathbf{y}^{\mathbf{k}}$ of total degree n or less substitute $y_i = f_i$ to get $g_{\mathbf{k}} = f_1^{k_1} \cdots f_r^{k_r} \in \mathbb{C}[x_1, \ldots, x_s]$.
- Set $\mathcal{G}_n(\phi)$ to be the Macaulay matrix of the polynomial list $[\{g_{\mathbf{k}}\}]$

Theorem: Using large enough n with high probability

$$\mathcal{G}_n(\mathbb{C}[\mathbf{y}]/\mathcal{J}) = \mathcal{G}_n(\phi)\mathcal{G}_n(\mathbb{C}[\mathbf{x}]/\mathcal{I})$$

Example: Implicitization of Parametric Curve

A parameterized curve (or surface) $\mathcal{Y}$ in $\mathbb{A}^3$ can be considered as the image of a polynomial map ϕ from $\mathbb{A}^1$ (resp. $\mathbb{A}^2$). But since $\mathcal{G}_n(\mathbb{C}[\mathbf{x}])$ is the identity matrix then $\mathcal{G}_n(\mathcal{Y}) = \mathcal{G}_n(\phi)$.

Consider the implicitly defined curve $\phi(t) = [t^2(t-2), (t-1)^2(t+1), t(t-1)(t-2)]$ which has been used (eg. Shen and Yaun) as a counterexample to the resultant method. In [D1] the local-global method was used to extract the equations for the 1-dimensional part. Here we can get these directly. The order 3 part of $\mathcal{G}_{10}(\phi)$ is sufficient, columns indexed by t^{j-1}. The equations are:

$$3 + 6x + 3x^2 - 4y - 3xy + y^2 - z - xz$$

$$x - x^2 - z + yz$$

$$3x + 3x^2 - xy - 3xz + z^2$$

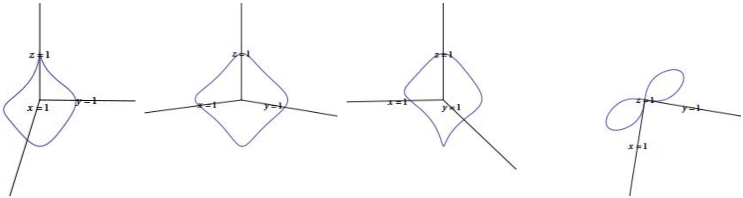
$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 4 & -4 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 3 & 1 & -3 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -4 & 8 & -5 & 1 & 0 & 0 & 0 & 0 \\
1 & -2 & -1 & 4 & -1 & -2 & 1 & 0 & 0 & 0 & 0 \\
0 & 2 & -5 & 2 & 4 & -4 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 4 & -12 & 13 & -6 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -8 & 12 & -6 & 1 & 0 \\
0 & 0 & 0 & 0 & 4 & -8 & 1 & 7 & -5 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 8 & -20 & 18 & -7 & 1 & 0 \\
0 & 0 & -2 & 5 & 0 & -9 & 6 & 3 & -4 & 1 & 0 \\
0 & 0 & 0 & -4 & 12 & -9 & -6 & 12 & -6 & 1 & 0 \\
0 & 0 & 0 & 0 & -8 & 28 & -38 & 25 & -8 & 1 & 0 \\
1 & -3 & 0 & 8 & -6 & -6 & 8 & 0 & -3 & 1 & 0 \\
0 & 2 & -7 & 5 & 9 & -15 & 3 & 7 & -5 & 1 & 0 \\
0 & 0 & 4 & -16 & 21 & -3 & -18 & 18 & -7 & 1 & 0 \\
0 & 0 & 0 & 8 & -36 & 66 & -63 & 33 & -9 & 1 & 0
\end{pmatrix}$$

Example: Graphing Implicit Space Curves

Plotting software may not handle implicit or contour plotting of space curves, especially if the curve is not a complete intersection. For a given projection $\phi : \mathbb{R}^3 \to \mathbb{R}^2$ we can find an implicit equation of the image in $\mathbb{A}^2$ which can then be plotted.

Consider the genus 2 space curve

$$\begin{cases} x + y - xz + yz \\ -x - 2x^2y - 2xy^2 - 2y^3 + xz + 2x^3z \\ -1 + 6x^2 + 8xy + 4y^2 - 4x^2z + z^2 + 2x^2z^2 \\ x^4 + xy + y^4 \end{cases}$$



Example: Weierstrass form for plane cubics

Almost any plane irreducible cubic $f(x,y)$ can be put in Weierstrass form $g(x,y) = y^2 - (x^3 + ax + b)$ for suitable $a, b \in \mathbb{C}$ by a projective linear transformation.

In fact there is an invertible 3×3 complex matrix A so that

$$(x_0, y_0) \in V(g) \Rightarrow A^{-1} \begin{bmatrix} x_0 \\ y_0 \\ 1 \end{bmatrix} = [x_1, y_1, z_1] \Rightarrow \left(\frac{x_1}{z_1}, \frac{y_1}{z_1} \right) \in V(f)$$

for all but a few points in $V(g)$. Moreover, if $f(x,y)$ is real and nonsingular then a, b, A can be chosen to be real.

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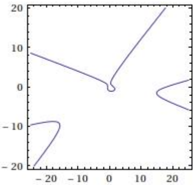
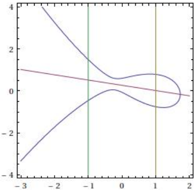
Consider the random real cubic in the next slide, the actual coefficients are 18 digit floating point numbers.

Example: Weierstrass curve continued

$$f(x,y) = -0.751079 - 0.454099x + 0.917319x^2 - 0.0494487x^3 + 0.646922y + 0.984103xy + 0.112161x^2y + 0.412702y^2 + 0.708919xy^2 - 0.779566y^3$$

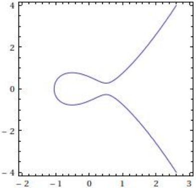


$$h(x,y) = -0.0268994 - 0.307028x - 0.387065x^2 + 0.335434x^3 - 0.345237y + 0.324363xy + 0.646742y^2$$



$$g(x,y) = y^2 - (0.33738 - 0.774147x + 1.x^3)$$

$$A = \begin{bmatrix} -0.282096 & 0.807945 & -0.172649 \\ -0.342732 & -0.274535 & 0.970200 \\ -0.82629 & 0.0138681 & -0.563074 \end{bmatrix}$$



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