

# Numerical Calculation of H-bases for Positive Dimensional Varieties

Barry H. Dayton  
Department of Mathematics,  
Northeastern Illinois University,  
Chicago, IL 60625  
bdayton@neiu.edu

April 21, 2011

## Abstract

A symbolic-numeric method for calculating an H-basis for the ideal of a positive dimensional complex affine algebraic variety, possibly defined numerically, is given. H-bases, introduced by Macaulay and later studied by Möller and Sauer [15, 16], for ideals  $\mathcal{I}$  in  $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \dots, x_s]$  are a numerical analog of a Gröbner basis with respect to a global degree ordering:  $f$  is an element of  $I$  of total degree  $n$  if and only if  $f$  is a  $\mathbb{C}$ -linear combination of polynomials of total degree  $n$  or less which are monomial multiples of members of the H-basis. The method uses the interplay of local and global duality.

Applications include finding the affine Hilbert function, factoring multivariable polynomials, finding branches of singular curves, finding equations for the union of varieties, and, most importantly, finding equations for components of reducible varieties given numerically.

## 1 Introduction

We introduce a symbolic-numeric method to find a H-basis, for an ideal  $\mathcal{I}$  in  $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \dots, x_s]$  provided the zero set  $V(\mathcal{I})$  is positive dimensional or has only a few known isolated zeros. This method, which makes use of both the original equations which locally define  $V(\mathcal{I})$  and numerically approximate points, generally works better than interpolation and requires fewer points. Although the method was developed to handle systems given numerically experiments on exact systems show that using machine precision (about 17 digits) numerical calculation one can generally approximate exact solutions to 9 or more digits for moderate sized systems.

These algorithms make use of the interplay between local and global duality via Macaulay and Sylvester matrices. The H-basis will generate the Sylvester matrices of the ideal for all orders which, like Gröbner bases, will provide a decision process for deciding if a polynomial is in the ideal and if so write it in terms of the basis, see Lemma 1. Möller and Sauer [16] show how, given an inner product on  $\mathbb{C}[\mathbf{x}]$ , normal forms may be calculated.

When calculating ideals of components of reducible varieties straight forward interpolation methods work poorly in the positive dimensional case and there are many solutions so picking a good generating system is not realistic. Sommese *et. al.* have suggested algorithms in [18, 2]. These require numerical irreducible decomposition and may return only a set theoretic basis whereas ours will generally be an ideal theoretic basis. They recently have proposed an algorithm [3] that will return a system with integer coefficients, when available, using the LLL or similar algorithms. But this method does not apply to the systems discussed in this paper.

H-bases are specifically introduced to handle affine systems and are numerical analogs of Gröbner bases given from global degree orderings, eg. `DegreeReverseLexicographic` of MATHEMATICA. Example 1 shows that in general H-bases are larger than arbitrary ideal generating sets but are often smaller than Gröbner bases. If the ideal of the system is homogeneous then any homogeneous basis for the ideal is an H-basis. Macaulay apparently viewed H-bases as generalizations of homogeneous bases, hence the *H*.

The author has implemented his algorithms using Macaulay arrays and SVD based numerical linear algebra to calculate the dual vectors. Various authors [13, 17, 21, 22] have used other methods to avoid the large matrices. These strategies may be useful in step 2 of the main algorithm `HBasis`, however in this paper we avoid these strategies for the following reasons: 1) the complexity of these methods is an unnecessary distraction from the subject at hand, 2) we want to emphasize the interplay of Macaulay and Sylvester arrays and 3) for most of our examples using sparse matrices makes the Macaulay method fast enough that, from a practical point view, the more sophisticated methods are not necessary and may actually be slower.

The method here is inherently numerical, there is no exact analog. For instance in Example 2 we factor a multivariate polynomial using a single numerical point on its variety as opposed to a-priori knowledge of the extension field over which the factors are defined. If one were to use exact arithmetic finding an analog of a Gröbner basis using a solution point is circular: a Gröbner basis is already needed. But, in the example cited above, one can find this point by randomly choosing complex values for all but one one variable and solving a univariate polynomial for the remaining variable. Other examples simply require knowledge of points of intersection with a linear space, if MATHEMATICA's `NSolve` can not handle it then software such as `PHCPACK` [20] can efficiently provide the solutions.

All computations in this paper were done with MATHEMATICA. Timings were taken using both MATHEMATICA 6 running on a Linux system and MATHEMATICA 7 running on Windows, to be conservative the slower MATHEMATICA 6 times are reported. The Mathematica software is available from the author on request.

## 2 Sylvester and Macaulay Arrays

Let  $F = [f_1, f_2, \dots, f_t]$  be a list of  $t$  polynomial functions in  $s$  variables  $x_1, \dots, x_s$  and  $\hat{\mathbf{x}} = (\hat{x}_1, \dots, \hat{x}_s)$  a point in  $\mathbb{C}^s$ .

For a non-negative integer list  $\mathbf{j} = [j_1, \dots, j_s]$  write  $|\mathbf{j}| = j_1 + j_2 + \dots + j_s$  and then  $\mathbf{x}^{\mathbf{j}} = x_1^{j_1} x_2^{j_2} \dots x_s^{j_s}$  is a monomial in  $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \dots, x_s]$  of *total degree*  $|\mathbf{j}|$ . The *total degree* of a

polynomial in these variables is the total degree of the largest monomial. Let  $(\mathbf{x} - \hat{\mathbf{x}})^{\mathbf{i}}$  denote  $(x_1 - \hat{x}_1)^{i_1} \dots (x_s - \hat{x}_s)^{i_s}$ . Assume a global degree ordering is given on the monomials of  $\mathbb{C}[\mathbf{x}]$ . As in [19, 7, 4] we use the differentiation operator

$$\partial_{\mathbf{x}^{\mathbf{j}}} \equiv \partial_{x_1^{j_1} \dots x_s^{j_s}} \equiv \frac{1}{j_1! \dots j_s!} \frac{\partial^{j_1 + \dots + j_s}}{\partial x_1^{j_1} \dots \partial x_s^{j_s}}. \quad (1)$$

where we write  $\partial_{\mathbf{x}^{\mathbf{j}}}[\hat{\mathbf{x}}](f)$  to indicate that we have applied the operator to function  $f$  and evaluated at point  $\hat{\mathbf{x}}$ .

The *Macaulay array of degree  $k$  at  $\hat{\mathbf{x}}$* ,  $\mathbf{M}(F, k, \hat{\mathbf{x}})$  is the  $n \binom{k+s-1}{s} \times \binom{k+s}{s}$  matrix with columns indexed by the differentials  $\partial_{\mathbf{x}^{\mathbf{j}}}$  for  $|\mathbf{j}| \leq k$  or, more commonly, just by the  $\mathbf{x}^{\mathbf{j}}$  ordered by a global degree ordering, we write  $\mathcal{X}_n$  for the ordered list of monomials of total degree  $n$  or less. In particular, the left hand column has index 1 for the evaluation functional. The rows will be indexed by the functions  $\mathbf{x}^{\mathbf{i}} f_\alpha$  for  $|\mathbf{i}| < k$ ,  $\alpha = 1, \dots, n$ . Again these will be grouped by degree  $|\mathbf{i}|$  and by monomial  $\mathbf{x}^{\mathbf{i}}$  in our ordering. In particular the first  $n$  rows are indexed by  $f_1, \dots, f_t$ .

*The entry in the row indexed by  $\mathbf{x}^{\mathbf{i}} f_\alpha$  and column indexed by  $\mathbf{x}^{\mathbf{j}}$  is  $\partial_{\mathbf{x}^{\mathbf{j}}}[\hat{\mathbf{x}}]((\mathbf{x} - \hat{\mathbf{x}})^{\mathbf{i}} f_\alpha)$ .*

The *Sylvester Array of degree  $k$* ,  $\mathbf{S}(F, k)$  of the list  $F = [f_1, \dots, f_t]$  is the submatrix of  $\mathbf{M}(F, k, \hat{\mathbf{0}})$ , where  $\hat{\mathbf{0}}$  is the origin, consisting of rows so that the total degree of the index polynomial  $\mathbf{x}^{\mathbf{i}} f_\alpha$  is less than or equal to  $k$ . The entry in the row indexed by  $\mathbf{x}^{\mathbf{i}} f_\alpha$  and column indexed by  $\mathbf{x}^{\mathbf{j}}$  is the coefficient of the monomial  $\mathbf{x}^{\mathbf{j}}$  in the polynomial  $\mathbf{x}^{\mathbf{i}} f_\alpha$ . In the Macaulay array the polynomials may be truncated, for the Sylvester array there is no truncation since only polynomials of total degree  $k$  or less are used.

Denote  $\mathcal{I} = \langle f_1, \dots, f_t \rangle$  to be the ideal generated by  $f_1, \dots, f_t$  in  $\mathbb{C}[\mathbf{x}]$  and put  $A = \mathbb{C}[\mathbf{x}]/\mathcal{I}$ . Then the row spaces of Macaulay arrays can be identified with a filtration of the maximal ideal of the local ring  $\mathcal{O}_{\hat{\mathbf{x}}}(A)$  while the row spaces of the Sylvester arrays filter the ideal  $\mathcal{I}$  so are associated with the global structure. But note while every element of  $\mathcal{I}$  is of the form of a finite sum  $g = \sum c_{\mathbf{i}} \mathbf{x}^{\mathbf{i}} f_\alpha$  the total degrees of some of the terms in this sum may be larger than the total degree of  $g$  due to some cancellation of terms. So not all the polynomials of total degree  $k$  or less may be in the sub-vector space of  $\mathcal{I}$  associated with  $\mathbf{S}(F, k)$ , see Example 1 below. If we want to include such polynomials we will use the notation  $\mathbf{S}(\mathcal{I}, k)$  for the Sylvester array of some set  $G$  so that the rowspace  $\mathbf{S}(G, k)$  contains the rowspace of  $\mathbf{S}(\{f\}, k)$  for all polynomials  $f$  of degree  $k$  or less in  $\mathcal{I}$ . This leads to the statement, and explanation of the choice of the letter ‘‘S’’, of our main definition:

**Definition 1** *Let  $\mathcal{I}$  be an ideal of  $\mathbb{C}[\mathbf{x}]$ . If  $\mathcal{B}$  is a set of polynomials in  $\mathcal{I}$  such that  $\mathbf{S}(\mathcal{B}, k) = \mathbf{S}(\mathcal{I}, k)$  for all  $k \geq 1$  then  $\mathcal{B}$  will be called an *H-basis* for  $\mathcal{I}$ . If no proper subset of  $\mathcal{B}$  is an *H-basis* we will call  $\mathcal{B}$  a *minimal H-basis*.*

For a polynomial  $f$  borrowing notation from [9] we write  $\text{jet}(f, n)$  for the polynomial consisting of all terms of  $f$  of degree  $n$  or less. We write  $\text{vjet}(f, n)$  for the vector consisting of all coefficients of  $\text{jet}(f, n)$  in the ordering given by  $\mathcal{X}_n$ , i.e.  $\text{vjet}(f, n) \mathcal{X}_n = \text{jet}(f, n)$ . In particular  $\text{vjet}(f, n)$  would be the row indexed by  $f$  in  $\mathbf{M}(\{f\}, n, \mathbf{0})$  so  $\mathbf{S}(\mathcal{I}, n)$  could be described by the property that  $\text{vjet}(f, n)$  is in the rowspace of  $\mathbf{S}(\mathcal{I}, n)$  for all  $f \in \mathcal{I}$  of total degree  $n$  or less.

The existence and main property of a minimal H-basis are given by

**Lemma 1** Let  $\mathcal{I}$  be an ideal of  $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \dots, x_s]$ . Then

- i) A finite H-basis for  $\mathcal{I}$  exists.
- ii) If  $\mathcal{B}$  is an H-basis for  $\mathcal{I}$  then given  $f \in \mathbb{C}[\mathbf{x}]$  of total degree  $n$ ,  $f \in \mathcal{I}$  if and only if  $\text{vjet}(f, n)$  is in the row space of  $\mathbf{S}(\mathcal{B}, n)$ .

**Proof:** For i) note that any Gröbner basis with respect to a global degree ordering [9, §1.2] contains an H-basis. ii) follows from the definition of  $\mathbf{S}(\mathcal{I}, n)$ . □

From Part i) finite minimal H-bases exist for each ideal of  $\mathbb{C}[\mathbf{x}]$ . Part ii) of Lemma 1 gives the recognition criteria for membership in  $\mathcal{I}$  and writing  $f \in \mathcal{I}$  in terms of the H-basis is then a matter of linear algebra.

**Example 1:** Consider the ideal  $\mathcal{I}$  of  $\mathbb{C}[x, y, u, v]$  generated by  $F = [x - y + uy + vy, -x + ux + vx + y, -1 + u + v + xy]$ . Then  $\text{rank } \mathbf{S}(F, 2) = 3$  but  $\text{rank } \mathbf{S}(\mathcal{I}, 2) = 5$  because of the additional degree 2 linearly independent polynomials  $-x^2 + y^2, 1 - 2u + u^2 - 2v + 2uv + v^2 - x^2$ . An H-basis consists of these 5 polynomials. The Gröbner bases given by MATHEMATICA using the DegreeReverseLexicographic ordering or by SINGULAR using the dp ordering require also the cubic  $x^3 - x$ . However  $\text{vjet}(x^3 - x, 3)$  is already in the row space of the Sylvester array of degree 3 given by the H-basis above so is not needed in the H-basis.

### 3 The Dual space of Differential Forms

#### 3.1 Local and Global Duality

Dual spaces on polynomial ideals have been studied by [15, 10, 8, 17] and also by [7, 6, 19], however the former use a global point of view while the latter take a local point of view with the exception of [19] who attempts to combine the two ideas. Since the main object of study by both groups are zero-dimensional rings the difference may not be apparent, however we are applying these to positive dimensional rings and the important distinction needs to be noted. We begin by briefly reviewing the two concepts of dual spaces in the papers cited above.

For the global point of view a *global differential functional* on  $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \dots, x_s]$  is a formal infinite sum  $\mathbf{D} = \sum_{\mathbf{k}} c_{\mathbf{k}} \mathbf{X}^{\mathbf{k}}$  where  $\mathbf{k}$  runs over all lists  $\mathbf{k} = [k_1, \dots, k_s]$  where the  $k_i$  are non-negative integers. The  $\mathbf{X}^{\mathbf{k}}$  are the functionals defined for each monomial  $\mathbf{x}^{\mathbf{j}}$  by

$$\mathbf{X}^{\mathbf{k}}(\mathbf{x}^{\mathbf{j}}) = \begin{cases} 1 & \text{if } \mathbf{j} = \mathbf{k}, \\ 0 & \text{if } \mathbf{j} \neq \mathbf{k}. \end{cases}$$

This extends  $\mathbf{D}$  to a linear functional over all polynomials in  $\mathbb{C}[\mathbf{x}]$ . If  $\mathcal{I}$  is an ideal of  $\mathbb{C}[\mathbf{x}]$  we say  $\mathbf{D}$  is in the *dual space* of  $\mathcal{I}$  if  $\mathbf{D}(f) = 0$  for all  $f \in \mathcal{I}$ . This makes  $\mathbf{D}$  a linear functional on  $\mathbb{C}[\mathbf{x}]/\mathcal{I}$ . By abuse of language we sometimes say  $\mathbf{D}$  is a *linear functional on*  $\mathcal{I}$ .

If  $\mathbf{D} = \sum_{\mathbf{k}} c_{\mathbf{k}} \mathbf{X}^{\mathbf{k}}$  we write  $\text{jet}(\mathbf{D}, p) = \sum_{|\mathbf{k}| \leq p} c_{\mathbf{k}} \mathbf{X}^{\mathbf{k}}$ . Then  $\text{vjet}(\mathbf{D}, p)$  will be the vector of coefficients  $c_{\mathbf{k}}$  indexed as in the Macaulay and Sylvester matrices. We have

**Lemma 2** Suppose  $F = [f_1, \dots, f_t]$  and  $\mathcal{I}$  is the ideal  $\mathbb{C}[\mathbf{x}]F$  generated by the  $f_i$ . Let  $\mathbf{D} = \sum_{\mathbf{k}} c_{\mathbf{k}} \mathbf{X}^{\mathbf{k}}$  be a differential functional. The following are equivalent.

- i)  $\mathbf{D}$  is a differential functional on  $\mathcal{I}$ .
- ii) For every  $n > 0$   $\text{vjet}(\mathbf{D}, n)$  is in the nullspace of  $\mathbf{S}(F, n)$ .
- iii) For every  $n > 0$   $\text{vjet}(\mathbf{D}, n)$  is in the nullspace of  $\mathbf{S}(\mathcal{I}, n)$ .

**Proof:** From the definitions i) is equivalent to iii) and iii) implies ii). But if  $g \in \mathcal{I}$  then the coefficient vector of  $g$  is contained in the row space of  $\mathbf{S}(F, d)$  for some  $d$  and hence  $\mathbf{D}(g) = 0$ . Thus ii) implies i). □

On the other hand a *local differential functional* on  $\mathbb{C}[\mathbf{x}]$  is a finite sum  $\mathbf{d} = \sum_{\mathbf{x}^{\mathbf{j}}} c_{\mathbf{x}^{\mathbf{j}}} \partial_{\mathbf{x}^{\mathbf{j}}}[\hat{\mathbf{x}}]$  where  $\mathbf{x}^{\mathbf{j}}$  runs over a finite set monomials in  $\mathbb{C}[\mathbf{x}]$ , each  $c_{\mathbf{x}^{\mathbf{j}}}$  is a complex number and  $\partial_{\mathbf{x}^{\mathbf{j}}}[\hat{\mathbf{x}}]$  was defined in (1). Local differential functionals are the differential functionals used in [19, 7, 6, 4].  $\hat{\mathbf{x}}$  is a point we will call the *center*, this point will act as a local origin, and local differential functionals will act on analytic functions as well as polynomials [6], a fact we will not use.

Given  $\hat{\mathbf{x}}$  in  $V(\mathcal{I})$ , the *local dual space at  $\hat{\mathbf{x}}$*  is the vector space of all local differentials  $\mathbf{d}$  centered at  $\hat{\mathbf{x}}$  such that  $\mathbf{d}(g) = 0$  for all  $g \in \mathcal{I}$ . A member of the local dual space at  $\hat{\mathbf{x}}$  will also be called a local differential functional on  $\mathbb{C}[\mathbf{x}]/\mathcal{I}$ , or on  $\mathcal{O}_{\hat{\mathbf{x}}}(\mathbb{C}[\mathbf{x}]/\mathcal{I})$  or on  $\mathcal{I}$  for an ideal  $\mathcal{I}$

The *order* of a local differential functional  $\mathbf{d}$ , written  $d = \text{ord}(\mathbf{d})$  the largest integer  $d$  so that  $c_{\mathbf{x}^{\mathbf{j}}} \neq 0$  for some  $|\mathbf{j}| = d$ . Again we can write  $\text{vjet}(\mathbf{d}, p)$  for the vector of coefficients  $c_{\mathbf{x}^{\mathbf{j}}}$  where  $|\mathbf{j}| \leq p$ , if  $p > \text{ord}(\mathbf{d})$  filling with 0's so that there is an entry for each index  $\mathbf{j}$  with  $|\mathbf{j}| \leq p$ . In contrast to Lemma 2, we have from [7, 6]

**Lemma 3** Suppose  $F = [f_1, \dots, f_t]$  is a list of polynomials and  $\mathcal{I}$  is the ideal of  $\mathbb{C}[\mathbf{x}]$  generated by the  $f_i$ . Then  $\mathbf{d}$  is a local differential functional at  $\hat{\mathbf{x}}$  if and only if  $\text{vjet}(\mathbf{d}, d)$  is in the nullspace of  $\mathbf{M}(F, d, \hat{\mathbf{x}})$  where  $d = \text{ord}(\mathbf{d})$ . Moreover,  $\text{vjet}(\mathbf{d}, p)$  will then be in the nullspace of  $\mathbf{M}(F, p, \hat{\mathbf{x}})$  for all  $p \geq d$ .

We note that although local differentials are finite sums, the vector space of all local differentials on  $\mathcal{I}$  at  $\hat{\mathbf{x}}$  may be infinite dimensional. In fact [6, Theorem 1] a point  $\hat{\mathbf{x}}$  in  $V(\mathcal{I})$  is an *isolated point* of  $V(\mathcal{I})$  if and only if the local dual space of  $\mathcal{I}$  is finite dimensional. On the other hand the global result, eg. [8, 17, 19], is that  $V(\mathcal{I})$  is finite if and only if the global dual space of  $\mathcal{I}$  is finite dimensional.

It is instructive to compare our distinction of local and global with the monomial orderings of [9]. Local orderings have notation such as  $\mathbf{1s}$ ,  $\mathbf{ds}$  where the “s” stands for series while global orderings are denoted by  $\mathbf{1p}$ ,  $\mathbf{dp}$  etc. where “p” stands for polynomial. The local dual differential functionals here make sense in the context of analytic functions or series [6] while the global dual differential functionals only apply to polynomials.

To sum up, local forms act at a point, global forms act on the whole variety, local forms are dual to the Macaulay arrays while global forms are dual to the Sylvester matrices. These differences become most apparent working with positive dimensional algebraic sets.

### 3.2 Local to Global

A key feature for polynomials is that we can convert local differentials to global. Given a local differential at a point  $\hat{\mathbf{x}} \in \mathbb{C}^s$  we have the following *change of center* formula:

$$\partial_{\mathbf{x}^{\mathbf{j}}}[\hat{\mathbf{x}}] = \sum_{\mathbf{i} \geq \mathbf{j}} \binom{i_1}{j_1} \hat{x}_1^{i_1-j_1} \cdots \binom{i_s}{j_s} \hat{x}_s^{i_s-j_s} \mathbf{X}^{\mathbf{i}} \quad (2)$$

where  $\hat{\mathbf{x}} = [\hat{x}_1, \dots, \hat{x}_s]$ ,  $\mathbf{i} = [i_1, \dots, i_s]$  and  $\mathbf{j} = [j_1, \dots, j_s]$ ,  $\mathbf{i} \geq \mathbf{j}$  means  $i_\alpha \geq j_\alpha$  for all  $1 \leq \alpha \leq s$  and, for this formula,  $x_i^0 = 1$  even if  $x_i = 0$ . Further,  $\hat{\mathbf{0}}$  always denotes the actual origin in that all coordinates of  $\hat{\mathbf{0}}$  are 0.

Note that the right hand side of (2) is a formal infinite sum so we do not get a local differential. But there are only finitely many terms of any degree so applying this to any polynomial gives only finitely many non-zero terms. If  $\mathbf{d} = \sum c_{\mathbf{j}} \partial_{\mathbf{x}^{\mathbf{j}}}$  linearity gives a global differential  $\mathbf{D} = \sum \xi_{\mathbf{i}} \mathbf{X}^{\mathbf{i}}[\hat{\mathbf{0}}]$ . But both give the same result when acting on any polynomial  $f \in \mathbb{C}[\mathbf{x}]$ . So if  $\mathbf{d}$  is in the local dual space to  $\mathcal{I}$  at  $\hat{\mathbf{x}}$ , then  $\mathbf{D}$  is in the global dual space to  $\mathcal{I}$ . We will write  $\mathbf{D} = \Gamma(\mathbf{d})$  and call  $\Gamma$  the *globalization operator*.

In particular, if  $\text{vjet}(\mathbf{d}, n)$  is in the nullspace of the Macaulay matrix  $\mathbf{M}(F, n, \hat{\mathbf{x}})$  then  $\text{vjet}(\mathbf{D}, n)$  is in the nullspace of the Sylvester matrix  $\mathbf{S}(F, n)$  for  $\mathbf{d} = \Gamma(d)$ . In fact, if  $\mathcal{I}$  is generated by  $F$  then Lemma 2 says that  $\text{vjet}(\mathbf{D}, d)$  is in the nullspace of  $\mathbf{S}(\mathcal{I}, d)$ .

Conversely if  $\mathbf{D}$  is in the nullspace of  $\mathbf{S}(F, n)$  it need not be in the nullspace of  $\mathbf{M}(F, n, \hat{\mathbf{0}})$ . In particular a local differential on  $\mathcal{I}$  is global but not conversely. So there is no global to local principle.

Note that (2) extends to a linear transformation on the local dual space at  $\hat{\mathbf{x}}$  to the global dual space of  $\mathcal{I}$ . Thus, given suitable bases for the space of all differentials at  $\hat{\mathbf{x}}$  and the global differentials, then there are matrices which give these transformations taking  $\text{vjet}(\mathbf{d}, n)$  to  $\text{vjet}(\mathbf{D}, n)$ . For example if  $s = 2$  and  $\hat{\mathbf{x}} = (1, 2)$  then with the monomial ordering  $\mathcal{X}_3 = \{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3\}$  this matrix is the  $10 \times 10$  lower triangular unit diagonal matrix

$$\gamma_{\hat{\mathbf{x}}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 3 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 4 & 1 & 2 & 2 & 0 & 0 & 1 & 0 & 0 \\ 4 & 4 & 4 & 0 & 4 & 1 & 0 & 0 & 1 & 0 \\ 8 & 0 & 12 & 0 & 0 & 6 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (3)$$

### 3.3 The Main Theorem

If  $\mathcal{I}$  is an ideal in  $\mathbb{C}[\mathbf{x}]$  we write  $\text{jet}(\mathcal{I}, n)$  for the  $\mathbb{C}$  vector subspace of  $\mathcal{I}$  of all polynomials of total degree at most  $n$ . As noted above this can be identified with the row space of  $\mathbf{S}(\mathcal{I}, n)$ . Our main theoretical result is

**Theorem 1** *Let  $\mathcal{I}$  be an ideal of  $\mathbb{C}[\mathbf{x}]$  and integer  $n > 0$  be given. Then there exist finitely many points  $\hat{\mathbf{p}}_j \in V(\mathcal{I}), j = 1, \dots, k$  such that for  $f \in \mathbb{C}[\mathbf{x}]$  of total degree no more than  $n$ ,  $f \in \text{jet}(\mathcal{I}, n)$  if and only if  $\mathbf{D}(f) = 0$  for each global vector  $\mathbf{D} = \Gamma(\mathbf{d})$  where  $\mathbf{d}$  is a local vector of order  $n$  or less at some  $\hat{\mathbf{p}}_j$ .*

**Proof:** Since  $f \in \mathcal{I}$  if and only if  $f$  lies in the local ideal  $\mathcal{I}_{\hat{\mathbf{x}}}$  for all  $\hat{\mathbf{x}} \in V(\mathcal{I})$  then strict local duality given in [8, 17] shows  $f \in \mathcal{I}_{\hat{\mathbf{x}}}$  if and only if  $\mathbf{d}(f) = 0$  for all local differentials at  $\hat{\mathbf{x}}$ . Thus  $f \in \mathcal{I}$  if and only if  $\Gamma(\mathbf{d})(f) = 0$  for all local differentials at all points of  $V(\mathcal{I})$ . Moreover if the total degree of  $f$  is no more than  $n$ ,  $f \in \text{jet}(\mathcal{I}, n)$  if and only if  $\Gamma(\mathbf{d})(f) = 0$  for all  $\mathbf{d}$  of order  $n$  or less. But since  $\text{jet}(\mathcal{I}, n) \subseteq \text{jet}(\mathbb{C}[\mathbf{x}], n)$  which is finite dimensional the restriction of the dual space to dual vectors operating on  $\text{jet}(\mathbb{C}[\mathbf{x}], n)$  is finite dimensional. Thus it has a basis consisting of finitely many  $\Gamma(\mathbf{d})$ , and these can be derived from finitely many points of  $V(\mathcal{I})$ .  $\square$

If  $P = \{\hat{\mathbf{p}}_1, \dots, \hat{\mathbf{p}}_k\}$  is a finite set of points such that the conclusion of Theorem 1 holds for  $\mathcal{I}$  and  $N$  we say  $P$  satisfies Theorem 1 for  $\mathcal{I}, N$ .

From a matrix point of view Theorem 1 says that if one constructs the column matrix  $\mathcal{D}_n = \mathcal{D}_n(\{\hat{\mathbf{p}}_1, \dots, \hat{\mathbf{p}}_k\})$  of all  $\text{vjet}\Gamma(\mathbf{d})$  of all local dual vectors  $\mathbf{d}$  of degree  $n$  or less at the points  $\hat{\mathbf{p}}_j$  then the Sylvester matrix  $\mathbf{S}(\mathcal{I}, k)$  is row equivalent to the left nullspace of  $\mathcal{D}_n$ . If  $N > n$  then if  $f$  is a polynomial in  $\mathcal{I}$  of degree  $n$  then  $\text{vjet}(f, N)$  is the vector of length  $\binom{N+s}{s}$  which agrees with  $\text{vjet}(f, n)$  in the first  $\binom{n+s}{s}$  places and is 0 in the remaining places. By Theorem 1  $\text{vjet}f, N$  is then in the left null space of  $\mathcal{D}_n = \mathcal{D}_N(\{\hat{\mathbf{p}}_1, \dots, \hat{\mathbf{p}}_k\})$  for any set of points  $(\{\hat{\mathbf{p}}_1, \dots, \hat{\mathbf{p}}_k\})$  of  $V(\mathcal{I})$ . But because of the zeros above degree  $n$  it is seen that  $\text{vjet}(f, n)$  is in the left nullspace of  $\text{vjet}(\mathcal{D}_N, n)$ , the submatrix of  $\mathcal{D}_N$  consisting of the first  $\binom{n+s}{s}$  rows. Conversely any vector in the left nullspace of  $\text{vjet}(\mathcal{D}_N, n)$  extends, by adding 0's at the end to a vector in the left nullspace of  $\mathcal{D}_N$  so corresponds to a polynomial of degree  $n$  in  $\mathcal{I}$  assuming  $\{\hat{\mathbf{p}}_1, \dots, \hat{\mathbf{p}}_k\}$  satisfies Theorem 1 for degree  $N$ . Thus we have shown

**Corollary 1** *Suppose points  $\{\hat{\mathbf{p}}_1, \dots, \hat{\mathbf{p}}_k\}$  satisfy the conclusion of Theorem 1 for  $N$  and  $n < N$ . Then the row space of the Sylvester matrix  $\mathbf{S}(\mathcal{I}, n)$  is the left nullspace of  $\text{vjet}(\mathcal{D}_N(\{\hat{\mathbf{p}}_1, \dots, \hat{\mathbf{p}}_k\}), n)$ .*

This corollary will provide the theoretical justification of the algorithms in this paper. How many points are needed cannot be easily quantified because all points of  $V(\mathcal{I})$  may not be equally good. To get a full H-basis for  $\mathcal{I}$  it can be seen that it is necessary to have at least one point from each component of  $V(\mathcal{I})$ . For fixed  $n$  as  $N$  grows the number of points needed per component decreases, in principle one point per component should be enough but computing costs grow quickly with  $n$  while adding new points is cheaper, so this idea is not pursued here.

Often if the degree of  $V(\mathcal{I})$  is  $d$  then  $d$  points are sufficient and intersecting  $V(\mathcal{I})$  by a random hyperplane gives general enough points. The result can be checked on additional random points and/or by numerical irreducible decomposition to see that in fact  $V(\mathcal{I})$  is defined, at least as a set theoretical intersection, by the resulting system.

## 4 The Hilbert Function

Kreuzer and Robbiano define an affine Hilbert function in [12, §5.6] which is easy to describe using our notation. We will call it the *global Hilbert function*  $\text{GHF}(n)$  to distinguish it from our *local Hilbert function* in [7, 6]. Given an ideal  $\mathcal{I}$  in  $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \dots, x_s]$

$$\text{GHF}(n) = \binom{n+s}{s} - \text{rank } \mathbf{S}(\mathcal{I}, n), \quad n > 0, \quad \text{GHF}(0) = 1 \quad (4)$$

Since we will be using Corollary 1 to calculate  $\mathbf{S}(\mathcal{I}, n)$  a more direct calculation of  $\text{GHF}(n)$  is given by

**Corollary 2** *If  $\{\hat{\mathbf{p}}_1, \dots, \hat{\mathbf{p}}_k\} \subseteq V(\mathcal{I})$  is a set of points for which the conclusion of Theorem 1 is true for  $N$  then for  $n \leq N$*

$$\text{GHF}(n) = \text{rank vjet}(\mathcal{D}_N(\{\hat{\mathbf{p}}_1, \dots, \hat{\mathbf{p}}_k\}), n)$$

**Example 2** [12, Example 5.6.2]: Consider  $\mathcal{I} = \langle xy, x^2 - y \rangle \subseteq \mathbb{C}[x, y]$ . Since  $V(\mathcal{I}) = \{(0, 0)\}$  it can be seen that Theorem 1 holds for  $\{(0, 0)\}$ . Thus the global dual space is the local dual space spanned by  $\partial_1, \partial_x, \partial_y + \partial_x^2$ . In [7, 6] this gives local Hilbert function  $1, 1, 1, 0, 0, \dots$ . The global duals are  $1, X, Y + X^2$  so  $\text{vjet}(\mathcal{D}_N(\{(0, 0)\}), 2)$  is the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}^\top$$

for  $N \geq 2$  where the rows are indexed by  $1, X, Y, X^2, XY, Y^2, \dots$ . So  $\text{vjet}(\mathcal{D}_N(\{(0, 0)\}), n)$  has rank 3 for  $n \geq 1$  and hence  $\text{GHF}$  is  $1, 3, 3, 3, \dots$

Further it is noted in [12] that  $\mathbb{C}[x, y]/\mathcal{I} \cong \mathbb{C}[x]/\langle x^3 \rangle$  which has  $\text{GHF}$   $1, 2, 3, 3, \dots$ . Thus for ideals  $\mathcal{J}$  in  $\mathbb{C}[\mathbf{x}]$  the global Hilbert function is an invariant of  $\mathcal{J}$  but not of the ring  $\mathbb{C}[\mathbf{x}]/\mathcal{J}$ .

□

We remark that there is a polynomial  $\text{GHP}(n)$  with rational coefficients which agrees with  $\text{GHF}$  for large  $n$  known as the *affine Hilbert polynomial*. The degree  $d$  of this polynomial is the Krull dimension of  $\mathbb{C}[x]/\mathcal{I}$ , see again [12] for details. If  $c_d$  is the leading coefficient of  $\text{GHP}$  we will call the integer  $d!c_d$  the *degree* of  $V(\mathcal{I})$  which agrees with the usual definition of degree of an irreducible variety.

## 5 The Algorithms

### 5.1 Computing $\mathbf{S}(\mathcal{I}, k)$

Let polynomials  $f_1, \dots, f_t$  be given, as in §3 we will write  $\mathcal{D}_N = \mathcal{D}_N(\hat{\mathbf{p}}_1, \dots, \hat{\mathbf{p}}_k)$  be the column matrix of all  $\text{vjet}(\Gamma(\mathbf{d}))$  where  $\mathbf{d}$  is of order  $N$  or less in the dual of the ideal  $\mathcal{I} = \langle f_1, \dots, f_t \rangle$  at some  $\hat{\mathbf{p}}_j$ .  $\mathcal{D}_N$  can be calculated by Macaulay matrices as in Lemma 2 and §3.2 or using strategies such as in [13, 17, 21, 22]. Our main algorithm is the following, here  $\mathcal{X}_n$  is a vector of column indices of  $\mathbf{S}(\mathcal{I}, n)$ .

**Algorithm HBasis:** Calculation of Sylvester Matrices.

- **Input:** Positive integers  $n \leq N$ , polynomials  $f_1, \dots, f_t$  defining  $V(\mathcal{I})$  and points  $\hat{\mathbf{p}}_1, \dots, \hat{\mathbf{p}}_j$  in  $V(\mathcal{I})$ .
- Calculate  $\mathcal{D}_N = \mathcal{D}_N(\hat{\mathbf{p}}_1, \dots, \hat{\mathbf{p}}_k)$ .
- Truncate this matrix to order  $n$ , i.e. take the first  $\binom{n+s}{s}$  rows  $\text{vjet}(\mathcal{D}_N, n)$  of  $\mathcal{D}_N$ .
- Find the nullspace of  $\text{vjet}(\mathcal{D}_N, n)$  as a row matrix  $S$ .
- **Output:** The matrix  $S$  or the set of polynomials  $S\mathcal{X}_n$ .

If  $\mathcal{I}$  and the points  $\hat{\mathbf{p}}_1, \dots, \hat{\mathbf{p}}_k$  satisfy Theorem 1 for order  $N$  then  $S = \mathbf{S}(\mathcal{I}, n)$ , and, if  $N$  is sufficiently large,  $\mathbf{S}(\mathcal{I}, N)$  will be a H-basis for  $\mathcal{I}$ . There are also applications of this algorithm when the conclusions of Theorem 1 are not satisfied, for example if  $\text{GHF}(n)$  calculated from  $\mathcal{D}_N$  is the correct value. Or see the next example.

**Example 3:** Consider the polynomial

$$\begin{aligned} f = & 2 + 3x^2 - 5x^4 + x^6 - 3y^2 - 16x^2y^2 + 8x^4y^2 - 5y^4 + \\ & 16x^2y^4 + 7y^6 + 9xz - 4x^3z - 2x^5z - 22xy^2z + \\ & 11xy^4z + 7x^2z^2 - 4x^4z^2 - 11x^2y^2z^2 + x^3z^3 \end{aligned}$$

This polynomial can be factored exactly over a complicated complex finite extension field of the rationals to give 3 real quadratic factors. However we merely have MATHEMATICA pick random complex values, approximately,  $\hat{y} = -0.59133 - 0.126784i$ ,  $\hat{x} = -0.2477 + 0.897805i$  which we substitute into  $f$  and solve the resulting one variable polynomial for  $\hat{z}$ . For  $\hat{\mathbf{p}}_1 = (\hat{x}, \hat{y}, 1.03993 + 5.73923i)$ , approximating the 17 digit machine numbers, we construct  $\mathcal{D}_7(\{\hat{\mathbf{p}}_1\})$  taking about 4.3 seconds. The Hilbert function with tolerance  $10^{-9}$  is 1, 4, 9, 16, 24, 30, 34, 36 which shows that Theorem 1 is not satisfied for  $\{\hat{\mathbf{p}}_1\}$  and  $N = 7$  but that the Sylvester matrix of order 2 obtained by the algorithm HBasis will contain the unique row  $\text{vjet}(g_1, 2)$  for a quadratic  $g_1$ . The time reported for this computation is zero since we are merely taking the nullspace of a tiny  $36 \times 10$  matrix. Thus from this single point  $\hat{\mathbf{p}}_1$  we obtain a potential factor of  $f$  in under 5 seconds. We repeat this for points  $\hat{\mathbf{p}}_2, \hat{\mathbf{p}}_3$  with the same  $\hat{x}, \hat{y}$  but  $\hat{z} = 0.232135 - 0.184349i$ ,  $0.395344 + 1.01247i$  respectively. Thus we obtain, after about 15 seconds, the following 3 factors with real coefficients, the small imaginary terms dropped, and

coefficients accurate to 9 digits

$$\begin{aligned} g1 &= -0.447561736 + 0.364721029x^2 + 0.812282765y^2 - 0.082840707xz \\ g2 &= -0.210422444 - 0.537639032x^2 - 0.327216587y^2 - 0.748061476xz \\ g3 &= 0.561390307 - 0.134811931x^2 - 0.696202238y^2 + 0.426578376xz \end{aligned}$$

Multiplying these as reported immediately above, then norming to have constant term 2, we obtain a polynomial whose difference from the exact  $f$  is about  $5 * 10^{-8}$  in the 2-norm.

## 5.2 Minimal S-Basis

Let  $\mathcal{X}_n$  be the vector of column indices of  $\mathbf{S}(\mathcal{I}, n)$  for an ideal of  $\mathbb{C}[x_1, \dots, x_s]$ . In theory it should hold that

$$\text{rowspan}(\mathbf{S}(\mathbf{S}(\mathcal{I}, n)\mathcal{X}_n, n+1)) \subseteq \text{rowspan}(\mathbf{S}(\mathcal{I}, n+1)) \quad (5)$$

From the point of view of numerical linear algebra this inclusion of row spaces could be tested by

$$\text{rank } \mathbf{S}(\mathcal{I}, n+1) = \text{rank} \begin{bmatrix} \mathbf{S}(\mathbf{S}(\mathcal{I}, n)\mathcal{X}_n, n+1) \\ \mathbf{S}(\mathcal{I}, n+1) \end{bmatrix} \quad (6)$$

where the second matrix is a block matrix and rank is measured by number of singular values above the current working tolerance. However using the algorithm **HBasis** of the last section (5) may not precisely hold. There are various strategies that may be used to overcome this issue, but in this paper we will use the simple, although not totally satisfying, strategy of loosening the working tolerance so that (6) holds. Since (6) almost holds the extra singular values of the test block matrix are still small so we can reset the tolerance to be slightly larger than the largest of these.

Given a matrix  $A$  which rowspace a subset of the rowspace of  $B$  there are many numerical algorithms which, to a tolerance  $\epsilon$ , will produce a submatrix  $B'$  of  $B$  such that all of the rows of  $B'$  are numerically independent of the rowspace of  $A$  and the block matrix  $\begin{bmatrix} A \\ B' \end{bmatrix}$  has the same rowspace the same rowspace as  $B$ . We let the reader choose her favorite and call the result  $B'$  a *complimentary matrix to  $A$  in  $B$* .

**Algorithm MBasis:** Calculation of Minimal H-basis.

- **Input:**  $\mathcal{D}_N(\hat{\mathbf{p}}_1, \dots, \hat{\mathbf{p}}_k)$  satisfying Theorem 1 for  $\mathcal{I}, N$  calculated to a tight working tolerance.
- Use **HBasis** algorithm to successively calculate  $\mathbf{S}(\mathcal{I}, 1), \mathbf{S}(\mathcal{I}, 2), \dots$  until one finds a non-empty Sylvester matrix  $\mathbf{S}(\mathcal{I}, n_0)$ , initialize  $\mathcal{B} = \mathbf{S}(\mathcal{I}, n_0)\mathcal{X}_{n_0}$
- For  $n = n_0, \dots, N - 1$  do
  - Calculate  $\mathbf{S}(\mathcal{I}, n+1)$  and check that (6) holds for  $n$ , if not increase  $\epsilon$  so that it does hold. If the new  $\epsilon$  is unacceptably large then quit, algorithm fails.
  - Let  $C$  be the complimentary matrix to  $\mathbf{S}(\mathbf{S}(\mathcal{I}, n)\mathcal{X}_n)$  in  $\mathbf{S}(\mathcal{I}, n+1)$ . Append  $C\mathcal{X}_{n+1}$  to  $\mathcal{B}$ .
- Output  $\mathcal{B}$ .

**Example 4:** Consider the plane curve defined by  $f = 8x^3 + x^4 + 12x^2y - 20xy^2 - x^2y^2 + 4y^3 + y^4$  which clearly has a singularity at the origin. We wish to desingularize this curve and use the information to find the tangent directions of the branches, which are irrational numbers. Since apriori we do not know the answer we use a random quadratic, approximately  $g = -.292846x + .999554y - .763056xz - .063694yz$ . At least locally the desingularized curve will be the component of the system  $f, g$  obtained by removing the unwanted multiple component  $x = y = 0$ . To illustrate algorithm MBASIS we will find a minimal H-basis describing this component.

Intersecting  $V(\langle f, g, \rangle)$  with a random hyperplane we obtain 8 points, three on the component  $x = y = 0$ . We calculate, starting with tolerance  $10^{-12}$   $\mathcal{D}_6(\{\hat{\mathbf{p}}_1, \dots, \hat{\mathbf{p}}_5\})$  which takes 6 seconds, the Hilbert function confirms we have a curve of degree 5, so Theorem 1 is satisfied. By HBASIS  $\mathbf{S}(\mathcal{I}, 1)$  is empty, where  $\mathcal{I}$  will be the ideal of the desingularization.  $\mathbf{S}(\mathcal{I}, 2)$  has rank 1, essentially the vjet of quadratic  $g$ , so we initialize  $B = \{g\}$ . Now  $\mathbf{S}(B, 3)$  has rank 4, and HBASIS finds  $\mathbf{S}(\mathcal{I}, 3)$  also to be of rank 4 with condition (6) satisfied, so  $\mathcal{B}$  will contain nothing of degree 3.  $\mathbf{S}(B, 4)$  has rank 10 while  $\mathbf{S}(\mathcal{I}, 4)$  has rank 14. Condition (6) is not satisfied so we raise tolerance to  $6 * 10^{-10}$  to make it true. We add the 4 new functions to  $\mathcal{B}$ . There is nothing new in  $\mathbf{S}(\mathcal{I}, 5)$  or  $\mathbf{S}(\mathcal{I}, 6)$  but we must raise the final tolerance to  $3 * 10^{-6}$  to keep (6) valid. Along the way we have calculated the Hilbert function of the ideal generated by  $\mathcal{B}$  using (4) and we appear to have our curve of degree 5 so we are satisfied that we have the correct  $\mathcal{B}$ .

Substituting  $x = y = 0$  into the 5 members of  $\mathcal{B}$  we find that all 5 have solutions  $\{-1.7727, 0.479915, 0.810186\}$  for  $z$  so we now know the points over  $(0, 0)$  in the curve  $V(\mathcal{B})$ , we can check that these points are all non-singular by checking the Jacobians, each is of rank 2. Projecting the null vectors of these Jacobians down to the plane we find the tangent lines to be  $y + .391382x$ ,  $y - 1.22713x$ ,  $y - 4.16425x$ . □

## 6 Applications

### 6.1 Components of Reducible Varieties

In both Examples 3 and 4 we illustrated the application that motivated this paper, finding a system of equations defining a subvariety of a reducible variety. In Example 3 we were fortunate that one point was enough, in Example 4 we were fortunate that many points on the desired component satisfied a simple condition,  $x \neq 0$  or  $y \neq 0$ . Another fortuitous situation is finding the top dimension locus of  $V(\mathcal{I})$  using MATHEMATICA's NSolve which, when presented with a non-zero dimensional system returns a set of points in the top dimensional locus of the variety. We illustrate with the following example.

**Example 5:** Consider the well known cyclic-4 example:

$$\begin{aligned} & x_1 + x_2 + x_3 + x_4 \\ & x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1 \\ & x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_1 + x_4x_1x_2 \\ & x_1x_2x_3x_4 - 1 \end{aligned}$$

Here use order 6 and the 4 points obtained by an attempted solution via `NSolve` and tolerance  $10^{-11}$ . It takes about one minute of computer time to calculate  $\mathcal{D}_6(\hat{\mathbf{p}}_1, \dots, \hat{\mathbf{p}}_4)$  from which we quickly get Hilbert function 1,3,6,10,14,18,22 which strongly suggests a locus consisting of a curve of degree 4. The algorithms `HBasis`, `MBasis` indicate an H-basis contains 2 linear and 1 fourth degree polynomial. Comparing Sylvester matrices we see that this complex H-basis is equivalent to the known H-basis  $\{x_1 + x_3, x_2 + x_4, -1 + x_1^2x_2^2\}$ .  $\square$

More generally one could use homotopy continuation numerical algebraic geometry software such as `PHCPACK` [20] or `BERTINI` [1] the latter of which automatically gives degree  $V(\mathcal{J})$  *witness points* on the component  $V(\mathcal{J})$ . If this is not enough `BERTINI` also has an automatic option to give more. `PHCPACK` can also give this information but requires a bit more effort from the user.

An alternative is to note that each point on a positive dimensional component of  $V(\mathcal{I})$  has infinitely many local dual differentials. These can be interpreted as tangency conditions. Thus while a function that is dual to just a few low order differentials at one point  $\hat{\mathbf{p}}$  need not be in the ideal of the component but should have a small residue near  $\hat{\mathbf{p}}$ .

**Example 6:** Consider the surface  $V(f)$  of Example 3. At the point  $\hat{\mathbf{p}}_1$  of that example calculate only  $\mathcal{D}_4(\hat{\mathbf{p}}_1)$ . Then the Hilbert function will be 1, 4, 9, 13, 15 thus the set of functions  $F$  `Sbasis` returns has dimension 3 more than  $\mathbf{S}(\mathcal{I}_1, 3)$  where  $\mathcal{I}_1$  is the ideal of that component. Because the numeric nullspace algorithm wants to return an orthogonal set of rows it is likely that none of the rows actually corresponds to an element of  $\mathcal{I}_1$ . If we perturb the  $x, y$  coordinates of  $\hat{\mathbf{p}}_1$  by a random complex number of a magnitude of, say, 0.3 to get  $\hat{x}_1, \hat{y}_1$  and then solve for  $\hat{z}_1$  then substitute the three points  $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$  in one of the functions in  $F$  one residue will be close to zero and the others far. The point with residue close to zero is quite likely in  $V(\mathcal{I}_1)$ .  $\square$

With care this technique will work more generally to produce additional points in the same component. When one has enough so that Theorem 1 holds for a small enough  $N$  then the ideal produced by H-BASIS should be the ideal of the component.

## 6.2 Union of Varieties

Suppose  $\mathcal{I}_1, \mathcal{I}_2$  are ideals in  $\mathbb{C}[\mathbf{x}]$  but perhaps otherwise related. If  $\mathcal{D}^1 = \mathcal{D}_N^1(\hat{\mathbf{p}}_1, \dots, \hat{\mathbf{p}}_j)$ ,  $\mathcal{D}^2 = \mathcal{D}_N^2(\hat{\mathbf{q}}_1, \dots, \hat{\mathbf{q}}_k)$  are the dual matrices for the two ideals both satisfying Theorem 1 for the same  $N$  then any  $f \in \mathcal{J} = \mathcal{I}_1 \cap \mathcal{I}_2$  will be dual to both, that is dual to the block matrix  $[\mathcal{D}^1 | \mathcal{D}^2]$ , and conversely. Thus applying the algorithms `HBasis`, `MBasis` will calculate a minimal H-basis for  $V(\mathcal{I}_1) \cup V(\mathcal{I}_2)$ .

**Example 7:** A classical result of Caley and Salmon says a non-singular cubic in complex projective space contains exactly 27 lines. Typically even if the equation of the cubic is real most of the lines are complex. In 1856 L. Schläfli showed how to construct 12 real lines, called a double 6, in a real cubic from which 15 additional real lines could be constructed. Hilbert and Cohn-Vossen [11] gave a popular account of this construction in theory. Your author has shown how to explicitly carry out this construction using numerical methods and has actually given in [5] a system of two homogeneous polynomials with numerical coefficients, one a cubic, such that the solution set is a real double 6. A text file with these equations,  $s_3, s_4$  is on the author's website. For this example we specialize to affine space.

There are a number of ways of recovering the equations of lines from this system, using the methods here one can find a single point (not on the intersection of two lines) and find the line through that point. Conversely from the equations of some of these lines plus one point on each one can use the method described in the paragraph preceding this example to find a system describing the union. For example the system

$$\begin{aligned} f = & 0.228117 + 0.596326x + 0.3437x^2 \\ & - 0.389689y + 0.291819xy + 0.110657y^2 \\ & - 0.17882z + 0.0615411xz - 0.109051yz - 0.421084z^2 \\ g = & 0.130703 + 0.467538x - 0.533375x^2 \\ & - 0.102931y - 0.228558xy + 0.0180106y^2 + 0.324017z \\ & + 0.502279xz - 0.127204yz + 0.20791z^2 \end{aligned}$$

describing 4 lines in a  $2 \times 2$  configuration [5] in the double 6 defined by  $s_3, s_4$  was constructed by this method in about 2.5 seconds.

## 7 Conclusion

By carefully distinguishing between local and global duals and exploiting the connection between the two we can work with positive dimensional ideals. Although further analysis and experience working with these method is necessary one can be optimistic that these methods will be useful working with algebraic sets given numerically and accurately.

## References

- [1] D. J. Bates, J. D. Hauenstein, A. J. Sommese and C. W. Wampler II, *BERTINI: Software for numerical algebraic geometry*, available at [www.nd.edu/~sommese/bertini](http://www.nd.edu/~sommese/bertini), Version 1.2, Nov. 2009.
- [2] D. J. BATES, J. D. HAUENSTEIN, C. PETERSON AND A. J. SOMMESE *Numerical Decomposition of the Rank-Deficiency Set of a Matrix of Multivariate Polynomials*, in L. Robbiano, J. Abbott (eds.) *Approximate Commutative Algebra*, Texts Monogr. Symbol. Comput., Springer-Verlag, 2009.

- [3] D. J. BATES, J. D. HAUENSTEIN, T. M. MCCOY, C. PETERSON AND A. J. SOMMESE *Recovering exactness from numerical data in algebraic geometry*, preprint.
- [4] B. H. DAYTON, *Numerical Local Rings and Local Solution of Nonlinear Systems*, Proceedings of SNC '07, ACM Press, pp. 79–86, 2007.
- [5] B. H. DAYTON, *Ideals of Numeric Realizations of Configurations of Lines*, in *Interactions of Classical and Numerical Algebraic Geometry*, D. J. Bates, G. Besana, S. Di Rocco and C. W. Wampler Editors, Contemp. Math. 496, AMS, 2009.
- [6] B. H. DAYTON, T. Y. LI AND Z. ZENG, *Multiple zeros of nonlinear systems*, to appear in *Mathematics of Computation*. Posted on Math. Comp. website [www.ams.org/journals/mcom](http://www.ams.org/journals/mcom) February 2011.
- [7] B. H. DAYTON AND Z. ZENG, *Computing the Multiplicity Structure in Solving Polynomial systems.*, Proc. of ISSAC '05, ACM Press, pp. 116–123.
- [8] J. EMSALEM, *Géométrie des points épais*, Bull. Soc. Math. France, 106 (1978), pp. 399–416.
- [9] G.-M. GREUEL AND G. PFISTER, *A Singular Introduction to Commutative Algebra*. Springer Verlag, 2002.
- [10] W. GRÖBNER, *Algebraische Geometrie II*, vol. 737 of Bib. Inst. Mannheim, Hochschul-taschenbücher, 1970.
- [11] D. HILBERT AND S. COHN-VOSSEN, *Geometry and the Imagination*, translated from the 1932 *Anschauliche Geometrie* by P. Nemenyi 1952. AMS Chelsea Publishing, 1999.
- [12] M. KREUZER AND L. ROBBIANO, *Computational Commutative Algebra 2*, Springer, 2005.
- [13] N. LI AND L. ZHI *Computing the multiplicity structure of an isolated singular solution: case of breadth one*, to appear in *Journal of Symbolic Computation*, 2010.
- [14] T. Y. LI AND Z. ZENG, *A rank-revealing method with updating, downdating and applications*, SIAM J. Matrix Anal. Appl., 26 (2005), pp. 918–946.
- [15] F. S. MACAULAY, *The Algebraic Theory of Modular Systems*, Cambridge Univ. Press, London, 1916.
- [16] H.M. MÖLLER AND T. SAUER, *H-bases for polynomial interpolation and system solving*, Adv. in Comp. Math. 12 (2000) pp. 335–362.
- [17] B. MOURRAIN, *Isolated points, duality and residues*, J. of Pure and Applied Algebra, 117-118 (1996), pp. 469-493.
- [18] A. J. SOMMESE, J. VERSHELDE AND C. W. WAMPLER, *Numerical decomposition of the solution sets of polynomial systems into irreducible components*, SIAM Journal on Numerical Analysis 38 (2001), pp. 2022-2046.
- [19] H. J. STETTER, *Numerical Polynomial Algebra*. SIAM, 2004.
- [20] J. VERSHELDE, *Algorithm 795: PHCpack: a general-purpose solver for polynomial systems by homotopy continuation*, ACM Transactions on Mathematical Software 25 (1999), pp. 251-276. Available at [www.math.uic.edu/~jan](http://www.math.uic.edu/~jan).

- [21] X. WU AND L. ZHI, *Computing the multiplicity structure from geometric involutive form*. Proc of ISSAC 2008, ACM Press, pp. 325–332.
- [22] Z. ZENG, *The closedness subspace method for computing the multiplicity structure of a polynomial system*, in *Interactions of Classical and Numerical Algebraic Geometry*, D. J. Bates, G. Besana, S. Di Rocco and C. W. Wampler Editors, Contemp. Math. 496, AMS, 2009, pp. 347–362.