Ideals of numerically generic realizations of configurations of Lines

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Generic realizations of configurations of lines as reducible projective varieties in $\mathbb{P}^n$ are constructed numerically. Examples are taken from projections of abstract seminormal configurations and the classical geometry related to Schläfli's double-six configuration. The ideal defining the homogeneous coordinate ring of a realization is calculated along with its Hilbert Function, identifying which examples are complete intersections. In particular a generic Schläfli double-six is constructed and shown to be the complete intersection of a cubic and quartic surface. Using the numerical algebraic geometry software Bertini, it is then possible to give explicit equations for all 27 lines on this cubic surface.

1 Introduction

This paper was motivated by the conference "Intersections of Classical and Numerical Algebraic Geometry" in honor of Andrew Sommese. I wished to contribute something of an aglebraic-geometric nature, i.e. concerning varieties of dimension greater than zero. As a result I decided to revisit a series of papers [5, 6, 7] written with Leslie G. Roberts in the early 1980's which is my most algebraic-geometric work. Although the focus there was on seminormality, we computed Hilbert polynomials and Hilbert functions of the realizations. We noted that the problem reduced to a finite calculation even though we did not have the computational capability of actually performing the calculation at that time.

I am connecting this to my current interest in numerical algorithms for calculating Hilbert functions. In [8] Zhonggang Zeng and I calculated the Hilbert function of an isolated multiple zero. Recent work of A. Leykin [12] suggests that the method will also give a calculation of specific values of the Hilbert function for higher dimensional varieties. Unlike the zero dimensional case it is not known in general how far one must calculate to infer the complete Hilbert function of a positive dimensional variety. Thanks to the results in [5, 6] there is a stopping criteria for the examples discussed in this paper. I give a modified version of our algorithm in [8] which takes advantage of the fact that I am working with homogeneous ideals. Further, I estimate the Hilbert series and Hilbert polynomial by finding upper bounds by reduction to the exact case. Most importantly, I then give a Hilbert function driven numerical algorithm for calculating a reduced basis for the intersection of homogeneous ideals in the case where the Hilbert Polynomial is known.

In [7, §2] we give a formal definition of an *abstract configuration of lines*, however for the purposes of this paper it is enough to describe configurations by pictures such as the double 6 in Figure 1. Here lines parallel in the picture represent skew, non-intersecting, lines as are lines that cross with a empty space at the intersection point, such as lines 1 and 7. But crossing lines, such as 1 and 8 will intersect.

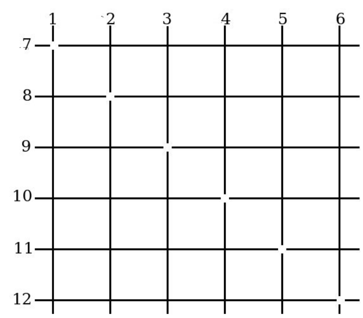

Figure 1: The Double-Six configuration

A realization of an abstract configuration of lines is a subvariety of real projective space $\mathbb{P}^n = \text{Proj}(\mathbb{R}[x_0, \ldots, x_n]), n \geq 2$, which is a union of one-dimensional lines, i.e. a union of varieties of the form $V(\langle f_1, \ldots f_{n-1}\rangle)$ where the f_i are linear homogeneous equations. These lines in the variety should have a 1-1 correspondence with those in the picture and corresponding lines in the variety intersect or are skew as are the originals in the picture. One could as easily work in complex projective space, but since almost all the configurations of interest can be represented in real projective space we will restrict to that for simplicity. For technical reasons noted below I will restrict myself to configurations with only two lines through each intersection point. In fact, most configurations will be sub-configurations of the double-six in Figure 1, but note that a realization of a subconfiguration may not be a subvariety of some realization of the original configuration.

The homogeneous coordinate ring of a realization of lines in $\mathbb{P}^n$ can also be viewed as the affine coordinate ring of a union of 2 dimensional planes in $\mathbb{A}^{n+1}$, real affine $n+1$ space. In fact this was L.G. Roberts and my original motivation for our study. We were examining the seminormality of a surface singularity but it was easier to describe the configurations as unions of projective lines. Unions of two dimensional planes are of interest, among other reasons, because they give some of the simplest examples of non-complete intersections, for example see [10, Ex 4.9,p. 224]. Most of the examples here are not complete intersections, but a few of the more interesting ones are, such as the double-six.

Besides avoiding the complexities of exact computation, using floating point arithmetic allows me to exploit the connection between *generic* in the classical sense and *numerically random*, see, for example, the discussion in Chapter 4 of [19]. I will obtain the examples from two sources, projections of abstract seminormal configurations [7] and the classical geometry related to Schläfli's Double-six configuratation in $\mathbb{P}^3$ [11]. In the first case the degree one part of a seminormal ring describing a configuration can be given by the nullspace of a certain kind of structured matrix. Replacing the parameters by random real numbers gives, with probability one, a generic example, I will call such an example *numerically generic*. By picking all, or part of, a random basis for this nullspace would then give, as described in Section 4 below, a random projection which gives the generic realization desired.

In the second case, Hilbert's recipe for construction of the double-six requires picking lines in "the most general possible position" [11, p.164]. This can be accomplished using numerically random vectors. A key step in Hilbert's construction, applied repeatedly, is to find the two lines which intersect each of four given skew lines not in *hyperboloidal position*, i.e not contained in a quadric surface. This requires finding a quadric containing 3 of the lines, solving a non-linear system to find the two points where the fourth line meets the quadric,and then finding the two lines through each of these points lying on the quadric. This last step can be accomplished by solving a non-linear system to find a random *witness* point on the line. Since most modern computer software will return approximate decimal solutions even to exact equations, it is almost essential to work numerically. Finding the quadric can be done by the ideal basis finding algorithm discussed above, and I will give a numerical algorithm in §5 for finding the lines through a point on a ruled surface. I report on BERTINI experiments which confirm the results of the earlier calculations. In particular I explain how one can use my ideal intersection algorithm together with BERTINI to give explicit equations for the remaining 15 lines in the cubic surface containing my generic double-six.

The numerical calculations will be done primarily using numerical linear algebra. Thus, in calculations, a line in $\mathbb{P}^n$ will often be represented by a $(n+1)\times(n-1)$ matrix of rank $n-1$, where the columns may be viewed as equations of the line, eg. the column $[a_{0i}, a_{2i}, \dots a_{1ni}]^\top$ represents the equation $a_{0i}x_0 + \cdots + a_{ni}x_n$. Thus two lines represented by matrices L_1, L_2 *intersect* if and only if the $(n+1)\times 2(n-1)$ block matrix $[L_1|L_2]$ is not of full row rank, in fact the intersection point is given in homogeneous coordinates by a non-zero vector in the nullspace of $[L_1|L_2]^\top$. For $n>2,\ 2(n-1)>(n+1)$ two distinct generic, i.e. random, lines will be skew. Unfortunately, using standard linear algebra software and floating point numbers, due to small roundoff errors, often a pair of lines will test skew, even if intended not to be. Thus all linear algebra must done relative to a fixed tolerance, i.e. in computing ranks all singular values below this tolerance must be considered zero. The experiments in this paper use 15 digit linear algebra with a tolerance of 10^{-8} except for the ideal calculation which works better with a tolerance of 10^{-7}.

The organization of this paper is as follows. Section 2 will describe the main algorithms: finding Hilbert functions, Hilbert series, and generators for the intersection of two ideals. Section 3 will briefly introduce the concept of seminormality and state results on Hilbert polynomials and functions from [5, 6, 7]. In section 4 I do experiments using *seminormal realizations* and section 5 will concern experiments related to Hilbert's construction of the double-six.

The goal of this paper is then twofold. First to demonstrate how a numerical point of view can be useful in constructing classical examples. Second to give results of experiments showing the utility of the algorithms in Section 2.

2 The Hilbert Function and Ideal Intersection Algorithms

In this section I will assume only that ideals I, J etc. are homogeneous ideals of $R = \mathcal{K}[x_1, x_2, \ldots, x_n]$ where $\mathcal{K}$ is the field of real or complex numbers. The *Hilbert function* $HF = HF_{R/I}$ of R/I is given by $HF(d) = \dim_{\mathcal{K}}(R/I)_{(d)}$ where for any graded ring or ideal A, $A_{(d)}$ will denote the space of homogeneous elements of A of degree d, $d = 0, 1, 2, \ldots$, the zero element being considered to have every degree.

By the Hilbert-Serre theorem [24, Chap. VII, §12] there is a polynomial function $HP = HP_{R/I}$ of d taking only integer values with $HF(d) = HP(d)$ for sufficiently large d. This is called the *Hilbert Polynomial* of R/I. See also [14, Chap. 5] but note that they use the notation HP for the Hilbert series, i.e. the formal power series I will denote by HS, $HS = \sum_{n=0}^{\infty} HF(n)t^n$. In this section I will give algorithms to calculate low degree terms of the Hilbert function, an upper bound for the Hilbert Series and generators for the intersection of two ideals when the Hilbert function is known.

2.1 Calculation of Hilbert Function

I begin with a lemma.

Lemma 1 *Let I be an ideal of $R = \mathcal{K}[x_1, \ldots, x_n]$ generated by homogeneous forms $f_1, \ldots f_k$ of various positive degrees. Then $I_{(d)} = \{\sum(c_{\mathbf{i}}\mathbf{x}^{\mathbf{i}} f_j)_{(d)} \bigm| \deg(\mathbf{x}^{\mathbf{i}}) < d\}$ where $c_{\mathbf{i}}\mathbf{x}^{\mathbf{i}} = c_{\mathbf{i}} x_1^{i_1} \ldots x_n^{i_n}, c_{\mathbf{i}} \in \mathcal{K}$ and for form $g, g_{(d)}$ is the degree d homogeneous part of g.*

Proof: Certainly I consists of all finite sums $\sum c_{\mathbf{i}}\mathbf{x}^{\mathbf{i}} f_j$. So $I_{(d)}$ consists of all sums $\sum(c_{\mathbf{i}}\mathbf{x}^{\mathbf{i}} f_j)_{(d)}$ But if $\deg \mathbf{x}^{\mathbf{i}} \geq d$ then all terms in that summand have degree greater than d so will contribute only 0 to this later summand. ∎

From the lemma, vector space information on $I_{(d)}$ can be calculated from what I prefer to call the *Macaulay array.* I think the name is appropriate as Macaulay used a similiar matrix, the *dialytic array* [15]. We used a similar matrix called the *Multiplicity* matrix in [8] and in [4] I used the term *Macaulay local array.* Here I am considering graded, rather than local, rings but it is exactly the same array as the local array so I am dropping the word "local".

As in [8] I use the differentiation operator

$$\partial_{\mathbf{x}^{\mathbf{j}}} \equiv \partial_{x_1^{j_1} \cdots x_s^{j_s}} \equiv \frac{1}{j_1! \cdots j_s!} \frac{\partial^{j_1+\cdots+j_s}}{\partial x_1^{j_1} \cdots \partial x_s^{j_s}}. \tag{1}$$

where I write $\partial_{\mathbf{x}^{\mathbf{j}}}[\hat{\mathbf{x}}](f)$ to indicate that the operator has been applied to function f and evaluated at point $\hat{\mathbf{x}}$. In this paper ideals are homogeneous so the proper base point is $\hat{\mathbf{x}} = \mathbf{0}$

and $\partial_{\mathbf{x}^{\mathbf{i}}}[0](f)$ simply picks out the coefficient of $\mathbf{x}^{\mathbf{i}}$ in an expanded representation of f. In [23] this matrix, in the reverse order is known as the *coefficient matrix*.

Let $F = [f_1, \ldots, f_k]^\top$ be a basis, i.e. set of homogeneous generators, for the homogeneous ideal I. The *local array of degree N at $\hat{\mathbf{x}}$*, $\mathbf{L}(F, N)$ is the $k\binom{N+n-1}{n} \times \binom{N+n}{n}$ matrix with columns indexed by the $\mathbf{x}^{\mathbf{j}}$ ordered, left to right, by the degree lexicographical ordering. In particular, the left hand column has index 1 for the monomial 1, this column should contain only zeros in this homogeneous context so may be omitted.

The rows will be indexed by the functions $\mathbf{x}^{\mathbf{i}} f_\alpha$ for $\deg(\mathbf{x}^b di) < d$, $\alpha = 1, \ldots, k$. Again these will be grouped by degree $\mathbf{x}^{\mathbf{i}}$ and by monomial $\mathbf{x}^{\mathbf{i}}$ in lexicographical order. In particular the first k rows are indexed by $f_1, \ldots, f_k$.

The entry in the row indexed by $\mathbf{x}^{\mathbf{i}} f_\alpha$ and column indexed by $\mathbf{x}^{\mathbf{j}}$ is

$$\partial_{\mathbf{x}^{\mathbf{j}}}[\mathbf{0}](\mathbf{x}^{\mathbf{i}} f_\alpha). \tag{2}$$

It can be seen that the row space of the submatrix of $\mathbf{L}(F, N)$ consisting of columns $\binom{d+n-1}{n}+1$ to $\binom{d+n}{n}$ is isomorphic to $I_{(d)}$ in an obvious way. Thus we have

Algorithm 1: Finding Hilbert Function

Given: $F = [f_1, \ldots, f_k]^\top$ homogeneous polynomials of various degrees in $K[x_1, \ldots, x_n]$ integer $N > 0$ and tolerance $\varepsilon > 0$.

- Construct Macaulay array $\mathbf{L}(F, N)$.
- For i from 1 to N do

  Find approximate rank to tolerance ε of submatrix from columns $\binom{i+n-1}{n} + 1$ to $\binom{i+n}{n}$ of $\mathbf{L}(F, N)$.

Output: Hilbert function of ideal $\langle f_1, \ldots, f_k \rangle$.

The most time consuming part of this algorithm is calculating the Macaulay array. Since, in the homogeneous case, the expansion is always about the origin the entries of the Macaulay array are just coefficients of the polynomials. Thus creating the array is just a bookkeeping problem. On a comuter algebra platform, however, one may find that using (2) is faster, i.e. calculation is faster than bookkeeping. In some of the examples below it is adequate, and more efficient, to estimate the Hilbert function by approximating the Hilbert series of an ideal given numerically by an exact monomial ideal. I do this next.

2.2 Approximating the Hilbert Series

The Hilbert function is a fairly coarse invariant of a ring, by [14, Th. 5.2.6] it is shown that the Hilbert series of an ideal is the Hilbert series of the leading term ideal with respect to a monomial ordering. This latter ideal is a monomial ideal so the Hilbert series can be calculated by standard software.

In the exact case a Gröbner basis computation is used to calculate the the leading term ideal, as this is written there is not a generally accepted numerical Gröbner basis algorithm. In [4] I obtained a Gröbner basis in the zero-dimensional local case using my *Approximate reverse row echelon form*, ARRREF, algorithm. A more sophisticated attempt in the general case using a variant of this algorithm is given in [18]. Thus I propose using the ARRREF. The problem is deciding what degree to use. I put that issue aside for the moment and describe the algorithm.

The ARRREF algorithm of [4] works as follows: Given an arbitrary complex matrix M and a tolerance ε, first use SVD to find the row space A to that tolerance. That is, in $M = USV^\top$ use the first r rows of $V^\top$ where r is the approximate rank, i.e. the number of singular values larger than ε. Then the pivot columns are those which are ε-independent from those to the right, this is where the word "reverse" comes in. This can also be checked by SVD or other approxi-rank software, e.g. [13]. Call the submatrix of the pivot columns P, this should be, and, in practice almost always is, a square matrix. Then the ARRREF is $P^{-1}A$. A good implementation returns a list of the column indicies as well as the matrix R.

It is then easily seen that the monomial indices of the pivot columns of the ARRREF form of the Macaulay array $\mathbf{L}(F,N)$ corresponds to the leading terms of polynomial combinations of degree less than or equal to N of the generators $f_1,\ldots,f_k$ of the ideal I. Thus the monomial ideal generated by these indices is contained in the full leading term ideal of I. Using too small a tolerance will possibly decrease the number of pivot columns, so one would have even a smaller monomial ideal. A smaller ideal gives a larger, termwise, Hilbert series so this gives the following lemma.

Lemma 2 *Let $F = [f_1,\ldots,f_k]^\top$ be a vector of homogeneous polynomials in $R = \mathcal{K}[x_1,\ldots,x_n]$, $I = \langle f_1,\ldots,f_k\rangle$, N an integer, and $\varepsilon > 0$. Let J be the monomial ideal generated by the indices of the ARRREF form of $\mathbf{L}(F,N)$. Then termwise, with high probability, $HS_{R/J} \geq HS_{R/I}$.*

The *high probability* refers to the fact that one cannot completely rule out some unusual numerical error producing false leading terms. In the experiments later in this paper, at least, this does not happen often.

Algorithm 2: Finding Hilbert Series

Given: $F = [f_1,\ldots,f_k]^\top$ homogeneous polynomials of various degrees in $K[x_1,\ldots,x_n]$ and tolerance $\varepsilon > 0$.

1. Let N be highest degree of some f_i.
2. Construct Macaulay array $\mathbf{L}(F,N)$.
3. Apply ARRREF algorithm to $\mathbf{L}(F,N)$ to get index monomials of pivot columns.
4. Apply standard Gröbner basis methods to obtain Hilbert series and polynomial of the monomial ideal obtained in step 3.
5. Check against known values of Hilbert function from Algorithm 1. If these don't check, increase N and go to step 2.

Output: Upper bound for Hilbert series and polynomial.

In the experiments in Sections 4,5 below knowing this bound is sufficient. In other cases one can use this and other information, such as a previously known Hilbert polynomial to make an educated guess of the Hilbert series.

Example 1: Consider the system of [4, Ex. 2] homogenized.

$$\begin{aligned}
f_1 =& 2x^2w - xw^2 - x^3 + z^3 \\
f_2 =& xw - yw - x^2 + xy + z^2 \\
f_3 =& xy^2z - x^2zw + x^3z
\end{aligned}$$

Algorithm 1 using $N = 6$ gives

$$HS = 1 + 4t + 9t^2 + 15t^3 + 20t^4 + 23t^5 + 24t^6 + \dots$$

Applying Algorithm 2 using $\mathbf{L}(F, 4)$ gives

$$\begin{gathered}
HS = 1 + 4t + 9t^2 + 15t^3 + 20t^4 + 25t^5 + 30t^6 + 35t^7 + \dots \\
HP(d) = 5d
\end{gathered}$$

This is not right so using $\mathbf{L}(F, 5)$ gives

$$\begin{gathered}
HS = 1 + 4t + 9t^2 + 15t^3 + 20t^4 + 23t^5 + 26t^6 + 29t^7 + \dots \\
HP(d) = 8 + 3d
\end{gathered}$$

This still gives a wrong value for $HF(6)$, using $\mathbf{L}(F, 6)$ gives

$$\begin{gathered}
HS = 1 + 4t + 9t^2 + 15t^3 + 20t^4 + 23t^5 + 24t^6 + 25t^7 + \dots \\
HP(d) = 18 + d
\end{gathered}$$

This fits the known Hilbert function, in addition we know from [4] that this system has a single one-dimensional linear component hence the coefficient of d in the Hilbert polynomial should be 1. So $HP(d) = 18 + d$ could be accepted as a good estimate of the correct Hilbert polynomial. In fact, since the original system here was exact this can be checked directly by standard Gröbner basis software and we find that the last Hilbert series and Hilbert polynomial are correct.

2.3 Intersection of Ideals

I now turn to the ideal intersection algorithm. Assume I, J are radical homogeneous ideals of $R = K[x_1, \dots, x_n]$. I wish to find a reduced basis of homogeneous polynomials for the intersection $I \cap J$. By *reduced basis* I mean

1. basis polynomials of the same degree are linearly independent.
2. no basis element is a linear combination of monomials multiplied by basis elements of smaller degree.

The algorithm below will calculate a reduced basis $f_1, \dots, f_k$ so that $< f_1, \dots, f_k >_{(d)} = (I \cap J)_{(d)}$ for $1 \le d \le N$ for some given N.

Algorithm 3: Finding reduced basis for degree d part of $I \cap J$, $1 \le d \le N$
Given: Basis F for I and G for J, integer N, variables $x_1, \dots, x_n$ and numeric tolerance ε.

- Construct $\mathbf{L}(F, N), \mathbf{L}(G, N)$.

- set H to empty vector.
- for d from 1 to N do
  - Find row space to tolerance ε for d-degree piece of $\mathbf{L}(F, N), \mathbf{L}(G, N)$ respecively as in Algorithm 1.
  - Find basis H' of vector space intersection of above row spaces to tolerance ε.
  - If H is empty set $H = H'$, break from loop, go to next d.
  - Find row space of d-degree piece of $\mathbf{L}(H, d)$, and calculate basis H'' of orthogonal complement to tolerance ε of this row space in span(H'). Add elements of H'' to H.

Output: H, the reduced set of generators of degree $\leq N$ for $I \cap J$.

To know that the output of Alorithm 3, $\langle f_1, \ldots, f_k \rangle$ is a complete set of generators for $I \cap J$. one will have to know additional information. In this paper one will know the Hilbert Polynomial of $I \cap J$ and at what point one can be sure the Hilbert function coincides with the Hilbert polynomial. Since the ideal $K = \langle f_1, \ldots f_k \rangle$ has Hilbert function no smaller than that of $I \cap J$ if we use Algorithm 2 to give an upper bound for the Hilbert function of K and this upper bound coincides with the known values of the Hilbert function of $I \cap J$ for $d > N$ then we have correct generators.

3 Seminormality

Although this paper is not about seminormality, I will use some results about seminormality from [3, 5, 6, 7]. Since many readers may not be familiar with this concept I will give a brief background and state the results needed later in this section.

Traditionally a non-singular curve has been called *normal*, and a *seminormal* curve was one with only normal crossings and singularities, such as a node. More complicated singularities, such as a cusp, were not seminormal. The modern study of seminormality dates to the two seminal papers by C.Traverso and R. G. Swan [21, 20] in 1970 and 1980 respectively. Loosely, from Traverso's point of view, ignoring technical hypotheses, the seminormalization of a ring A is the largest overring S in the normalization of A with $\mathrm{Spec}(S) = \mathrm{Spec}(A)$. Thus, in some loose sense, S is the most generic ring giving the same geometry as A. This will be my point of view in this paper. Swan, on the other hand, gave an arithmetic condition applicable to all commutative rings: A is seminormal if given $a, b \in A$ with $a^3 = b^2$ there is a unique $c \in A$ with $c^2 = a$ and $c^3 = b$. In both cases seminormal rings satisfied the homotopy condition $\mathrm{Pic}(A) = \mathrm{Pic}(A[t_1, \ldots, t_n])$ for all $n \geq 1$. Although this last condition was my motivation for studying seminormality in the 1980's I will not mention this again here. It should be noted that it follows immediately from Swan's criteria that any normal ring and any UFD is seminormal, in particular the polynomial rings $\mathcal{K}[x_1, \ldots, x_n]$ are seminormal for any field $\mathcal{K}$.

Most important for the present paper is F. Orecchia's theorem for reducible rings [17]. This says, in particular, that if $A = R/(\cap_{i=1}^{N} I_i)$ with each R/I_i seminormal and each $R/(I_i + I_j)$ reduced, i.e. no non-trivial nilpotent elements, for all $i \neq j$, then the seminormalization S of A is given by

$$S = \{(a_1, \ldots, a_N) \in \prod_{i=1}^{N} R/I_i \,\big|\, a_i \equiv a_j \bmod (I_i + I_j) \text{ all } i, j\} \tag{3}$$

A simple self-contained proof, with the exception of one easy lemma each from [20, 5], of this theorem from Swan's point of view is given in [3]. We, L. G. Roberts, C. A. Weibel and I, have called this result the *Chinese Remainder Theorem* because if, with notation as above, $R/(\cap_{i=1}^{N} I_i)$ is seminormal then if $a_1, \ldots, a_N \in R$ so that $a_i \equiv a_j \bmod (I_i + I_j)$ for all i, j there exists $a \in R$ with $a \equiv a_i \bmod I_i$ for all i.

There is one point of caution that must be mentioned. The ring S above may not be generated by elements of degree 1, even when A is the coordinate ring of a variety, examples will be given below. In this case S will not be the homogeneous coordinate ring of a projective variety. Thus the seminormalization of a union of lines in $\mathbb{P}^n$ may not be a union of *straight* lines in some $\mathbb{P}^r$. See the comment on [6, p. 432].

I now turn to the case where $R = \mathcal{K}[x_0, \ldots, x_n]$ and $a = R/(\cap_{i=1}^{N} I_i)$ are the homogeneous coordinate rings of the realization of an abstract configuration of lines with seminormalization S. I will state three essential theorems about the Hilbert function and polynomial of the rings A, S from [5, 6].

Theorem A: *Suppose the lines through each intersection point a union of lines have independent directions. Then* $\dim S/A < \infty$. *In particular* $HP_A = HP_S$ *when there are only two lines through each intersection point of the configuration.*

This follows from [6, Cor. 4.2]. The construction of S in (3) is quite combinatorial. As will be shown below this is not enough to determine the Hilbert function of S but, under the hypothesis above, it is enough to determine the Hilbert polynomials.

Theorem B: *Let S be the seminormalization of the union of N lines. Then*

i) $HP_S(d) \leq H_S(d)$ *for all* $d \geq 1$ *and if* $HP_S(d_0) = H_S(d_0)$ *for some* d_0 *then* $HP_S(d) = H_S(d)$ *for all* $d \geq d_0$.

ii) $HP_S(d) = H_S(d)$ *for all* $d \geq N - 2$.

iii) If only 2 lines meet at each intersection point and there are m intersection points then $HP_S(d) = N(d+1) - m$.

This is just [6, Th. 1.5]. In general the formula for the Hilbert polynomial is slightly more complicated when there are more lines through some points. Since the hypotheses of only two lines through a point simplifies matters, I will restrict to this case throughout this paper. Several examples below, for example the double-five subvariety of the double-six, have the property that $H_A(d_0) = H_S(d_0)$ for a particular d_0 but $H_A(d) \neq H_S(d)$ for some $d > d_0$. Of course $H_A(d) \leq H_S(d)$ for all $d \geq 1$. Moreover, it follows from the proof of [5, Th. 20] that

Theorem C: *Let A be the homogeneous coordinate ring of a union of N lines with seminormalization S. If* $H_A(d_0) = H_S(d_0)$ *for some* $d_0 \geq N - 2$ *then* $H_A(d) = H_S(d)$ *for all* $d \geq d_0$.

Putting these theorems together with the results of §2 gives

Theorem 1 *Let* $A = R/I,\ R = \mathbb{R}[x_0, \ldots, x_n]$ *be the homogeneous coordinate ring of a realization in* $\mathbb{P}^n$ *of an abstract configuration of N lines so that only two lines pass through any intersection point and there are m total intersection points. If d_0 is the smallest integer*

$d_0 \geq N-2$ such that $HF_A(d_0) = N(d_0+1) - m$ then $HF_A(d) = N(d+1) - m$ for all $d \geq d_0$. Further, if $f_1, \ldots, f_k \in I$ and $d_0 \geq N-2$ are such that the ideal $K = \langle f_1, \ldots, f_k \rangle$ satisfies $HF_{R/K}(d) = HF_A(d)$ for $d < d_0$ and the approximate Hilbert polynomial from Algorithm 2 is $HP_{R/K}(d) = N(d+1) - m$ then with, high probability, $K = I$.

By *complete intersection* I mean *strict* complete intersection, that is the ideal of a realization of lines in $\mathbb{P}^n$ is generated by $n-1$ elements. A result of Geramita and Weibel [9] says a connected union of lines, i.e. the diagram is connected, is seminormal if and only if it is Cohen-Macaulay. Since it is known [16, p. 171] that locally a complete intersection is Cohen-Macaulay, a necessary condition for complete intersections is that the homogeneous coordinate ring be seminormal. Examples in this paper confirm this result, but show the converse is not true.

4 Projections of abstract seminormal realizations

Starting from an abstract configuration of lines one can construct abstract seminormal realizations as in our discussion of Orecchia's theorem above. In general these do not correspond to seminormalizations of unions of lines in some $\mathbb{P}^r$ because they may not have enough elements of degree 1. Examples will be given where it is possible to *project* the abstact seminormal realizations to actual realizations of the configuration in some $\mathbb{P}^r$. I then will calculate the Hilbert functions and find the ideal.

The ring S described by the "Chinese Remainder Theorem" in the previous section can be presented by exact sequences [7, (2.8)]

$$0 \to S_{(d)} \to \prod \mathbb{R}[x_i, y_i]_{(d)} \xrightarrow{\theta_{(d)}} \prod \mathbb{R}[t_j]_{(d)} \qquad (4)$$

where the product of the $\mathbb{R}[x_i, y_j]$ is over all lines and the product of $\mathbb{R}[t_j]$ is over all intersection points. As in the last section the standing assumption will be that only two lines pass through an intersection point. Note that the two right hand terms of (4) can be identified with $\mathbb{R}$-vector spaces so the maps $\theta_{(d)}$ can be represented by matrices with certain structures.

The goal here is to start with an abstract configuration and construct seminormal realizations so I work backwards, mainly describe a matrix which should work for $\theta_{(1)}$ and give criteria that will imply that $\theta_{(1)}$ is a map for the degree one part of ring S as in (4).

Let $\mathcal{C}$ be an abstract configuration of lines and consider the matrix $M(\theta)$ as given in [6, p. 78]. There are two columns, bicolumns, labeled x_i, y_i, for each line i and one row, labeled t_j for each intersection point. Each row will be zero except for two pairs $(a_{j1}, b_{j1}), (a_{j2}, b_{j2})$ in the pair of bi-columns corresponding to the two lines intersecting that intersecting point. At least one coordinate of each pair must be non-zero. Additionally, all pairs in a bi-columns x_i, y_i must be independent as vectors in $\mathbb{R}^2$. Consider these pairs as homogeneous coordinates of the intersection points relative to some local homogeneous coordinate system on the line. In particular, the first two, from top, can always be $(1, 0)$ and $(0, 1)$, i.e. these first two points can determine the homogeneous coordinate system.

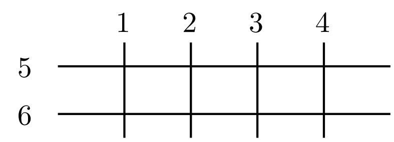

Figure 2: The 2-4 configuration

For example, consider the 2-4 configuration in Figure 2.

Then $M(\theta)$ could be given by Figure 3 where blank spaces indicate mandatory zeros.

	x_1	y_1	x_2	y_2	x_3	y_3	x_4	y_4	x_5	y_5	x_6	y_6
t_1	1	0							1	0		
t_2	0	1									1	0
t_3			1	0					0	1		
t_4			0	1							0	1
t_5					1	0			a_5	b_5		
t_6					0	1					a_6	b_6
t_7							1	0	a_7	b_7		
t_8							0	1			a_8	b_8

Figure 3: $M(\theta)$ for the 2-4 configuration.

In Figure 3 $a_5, \ldots, a_8,\ b_5, \ldots, b_8$ are parameters, some of which can be eliminated as seen below. If these parameters are replaced by random real numbers, all non-zero, then we will have defined $S_{(1)}$ in a *numerically generic seminormal realization* of our configuration. Moreover, we can get a basis by finding the approximate nullspace of the matrix $M(\theta)$ with respect to a tolerance ε. In this particular case the 8×8 submatrix of the first 8 columns is the identity so the nullspace, $\mathbf{N}$ will be of dimension 4. If $[c_1, \ldots, c_{12}]^\top$ is a vector in $\mathbf{N}$ then it can be interpreted as the element

$$(c_1 x_1 + c_2 y_1,\ c_3 x_2 + c_4 y_2, \ldots,\ c_{11} x_6 + c_{12} y_6) \in S_{(1)}. \tag{5}$$

In this case $S_{(1)}$ is spanned by 4 independent elements of this type. Here, and below, I will identify the 12×4 matrix with the vectors $[c_1, \ldots, c_12]^\top$ and its columnspace, the nullspace, and the space $S_{(1)}$.

Pick a basis for $S_{(1)}$, perhaps the columns of $\mathbf{N}$, and label the basis elements $z_0, \ldots, z_r$. Next take a subset of this basis $z_0, \ldots, z_n$ where $2 \leq n \leq r$. Thinking of the z_j as members of the nullspace of $M(\theta)$ view them as linear operators on the rows of $M(\theta)$. Restricting to just the two columns labeled above x_i, y_i then there can be at most 2 independent linear operators so there are at least $(n+1)-2 = n-1$ independent linear equations in $z_0, \ldots, z_n$, for $2 \leq n \leq r$ of the form $a_{j0} z_0 + aj1 z_1 + \ldots a_{jn} z_n$ which vanish on columns x_i, y_i. Let $\breve{I}_i$ be the linear subspace spanned by these linear forms of the vector space with basis $z_0, \ldots, z_n$.

I can now paraphrase [7, Th. 3.4 and Cor. 3.7].

Theorem D: *Let $M(\theta)$ be a structured matrix for the abstract configuration C of N lines. Construct the subspaces $\breve{I}_i$, $i = 1, \ldots, N$ as in the above discussion for a given n, $2 \leq n \leq r$*

where $r+1$ is the dimension of the nullspace of $M(\theta)$. Suppose

1. $\dim_{\mathbb{R}} \breve{I}_i = n-1$ *for all* i.
2. $dim_{\mathbb{R}}(\breve{I}_i + \breve{I}_j) = n$ *if lines* i, j *intersect in* C.
3. $dim_{\mathbb{R}}(\breve{I}_i + \breve{I}_j) = n+1$ *if lines* i, j *are skew in* C.

Next let I_i be the ideal of $R = \mathbb{R}[z_0, \ldots, z_n]$ generated by $\breve{I}_i$, and let $A = R/\cap_{i=1}^{N} I_i$. Then Proj(A) is a union of N straight lines in $\mathbb{P}^n$ with configuration C and $S_{(1)}$ can be identified with the one dimensional forms of the seminormalization S of A.

Examples below show the hypotheses of Theorem D are almost always satisfied so long as $n+1 \geq 2N - m \geq 4$ where m is the number of intersection points in C. When the hypotheses are satisfied by $M(\theta)$ I will call Proj(A), or just A, a realization of C and the ring S an abstract seminormal realization of C. In particular, when $n < r$ we called Proj(A) a *projection* of S, see [7, Th. 3.2] for an explaination of this terminology. If some of the entries of $M(\theta)$ are randomly chosen floating point numbers, or if the z_i are a random linear combination of the entries of $\mathbf{N}$, then the above discussion constructs a *numerically generic projection of an abstract seminormal realization of the configuration of lines C.*

Example 2: I continue with the 2-4 configuration of Figures 2,3. For the purposes of exposition I will replace the parameters in Figure 3 with integers and use exact, rather than my numerical, methods to obtain exact answer. It will be seen below that the result is not generic!

So replace the lower right 4×4 block of $M(\theta)$ in Figure 3 with

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

Then a 12×4 matrix with **rows** spanning the nullspace is

$$\mathbf{N} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & -1 & 0 & -2 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & -1 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Normally the nullspace would be represented as columns, but I give the transpose here to save space. Let z_0, z_1, z_2, z_3 be the rows of $\mathbf{N}$ and $n = r = 3$ to project the abstract seminormal configuration to $\mathbb{P}^3$. Continuing to work computationally I first calculate the "lines" as matrices, let $\mathbf{n}_i$ be the 2×4 matrix whose columns form span the left nullspace of columns $2i-1, 2i$ of the matrix $\mathbf{N}$ above. Then

$$\mathbf{n}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \mathbf{n}_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{n}_3 = \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}, \mathbf{n}_4 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \mathbf{n}_5 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{n}_6 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

By inspection each of these matrices is of rank $2 = n-1$. One then easily checks conditions 2,3 of Theorem D by calculating the rank of the matrices $[\mathbf{n}_i | \mathbf{n}_j]$ for $1 \leq i \leq j \leq 6$. Hence the lines in $\mathbb{P}^3$ given as nullspaces of these matrices give a realization of the 2-4 configuration, in fact the ideals are

$$I_1 = \langle x, z\rangle, I_2 = \langle y, w\rangle, I_3 = \langle -x+y, -z+w\rangle, I_4 = \langle x+y, z+w\rangle, I_5 = \langle y, x\rangle, I_6 = \langle z, w\rangle$$

Now using Gröbner basis methods rather than Algorithm 3 I find genenerators for the ideal $\cap_{i=1}^{6} I_i$ as

$$zy - wx,\ -x^3w + y^2xw,\ -x^2wz + w^2xy,\ -z^2wx + w^3x \tag{6}$$

Note that this realization is contained in the quartic $zy - wx$ which is not generic for the 2-4 configuration. Further since $HF_S(2) = HP_S(2) = 10 \neq HF_A(2) = 9$ this realization is not seminormal as predicted by [5, Example 7].

4.1 Numerically generic realizations of 2-p Configuration

The 2-p configuration, like the particular case of the 2-4 configuration, has 2 horizontal lines and p skew vertical lines. In this subsection I find realizations and calculate the ideal for $p = 1, \ldots, 7$. I start with a matrix with the structure of Figure 3 but with $2p$ rows and $2p+4$ columns. In all cases the leftmost $2p \times 2p$ submatrix is the identity so $M(\theta)$ will always have rank $2p$ and the nullspace will be of rank 4. Thus the realizations will always be in $\mathbb{P}^3$. For $p = 1, 2$ there will be no parameters to choose, but for $p \geq 3$ there will be $4(p-2)$ choices. These choices will be random non-zero real numbers $r, -1 < r < 1$.

Table 1 gives the results. The first column is p, the second is the Hilbert function as a sequence, $\to$ indicates the sequence continues following the Hilbert polynomial which is given. In the fourth column the degrees of the reduced basis are given, since there may be lots of generators I write, say, $2^5, 3^2$ to indicate 5 generators of degree 2 and two generators of degree 3. And in the last column the letter "S" indicates the realization is seminormal, and "CI" indicates complete intersection.

Table 1: The 2-p configurations.

p	Hilbert Function	Basis	Properties
1	$1, 4, 7, \to 3d + 1$	2^3	S
2	$1, 4, 8, 12, \to 4d$	2^2	S,CI
3	$1, 4, 9, 14, \to 5d - 1$	$2, 3^2$	S
4	$1, 4, 10, 16, 22, \to 6d - 2$	3^4	S
5	$1, 4, 10, 18, 25, \to 7d - 3$	$3^2, 4^2$	
6	$1, 4, 10, 20, 28, \to 8d - 4$	4^7	
7	$1, 4, 10, 20, 31, 40, \to 9d - 5$	$4^4, 5^2$	

Note that the numerically generic 2-4 here has a different Hilbert function than that of Example 2. For instance, $HF(2) = 10$ so this realization of the 2-4 is not contained in a quadratic. On the other hand, it is seminormal. Note also that each of the above realizations come from different projections, the $2-5$ realization may not be not be a subvariety of the $2-7$, for example.

The necessity of 7 fourth degree equations for the 2-6 configuration can be read off the Hilbert function. $HF(4) = 28$ but there are 35 independent four variable quartics. Since there are no equations of lower degree the ideal must contain a 7 dimensional space of fourth degree equations.

Calculation of the 2-7 configuration is at the limit of of the current implementation of my algorithms. Some runs of Algorithm 3 gave incorrect results due to numerical errors. The problem is in the reducing stage of the algorithm which is the most numerically sensitive. In this example the errors were highlighted by failure of the inequality in Lemma 2 in degrees where the Hilbert function is known by Algorithm 1.

4.2 Projections of the Double-three.

The Double-3 configuration may be thought of as lines 1,2,3,7,8,9 of Figure 1. Alternatively the configuration is a hexagon where each side is skew to all but the two adjoining sides. There are, from either point of view, 6 lines and 6 intersection points, with only two intersection points on each line. Thus $M(\theta)$ would have no parameters and look like

	x_1	y_1	x_2	y_2	x_3	y_3	x_4	y_4	x_5	y_5	x_6	y_6
t_1	1	0							1	0		
t_2	0	1									1	0
t_3			1	0			1	0				
t_4			0	1							0	1
t_5					1	0	0	1				
t_6					0	1			0	1		

Figure 4: $M(\theta)$ for the Double-3 configuration.

The rank is 6, exactly, and the nullspace has dimension 6. I introduce some randomization by multiplying the 12×6 matrix defining the nullspace by a random orthogonal matrix to get $\mathbf{N}$. Now I project onto $\mathbb{P}^3, \mathbb{P}^4$ and $\mathbb{P}^5$ by choosing the first 4,5 and 6 columns of $\mathbf{N}$ respectively. Table 2 uses the conventions used in Table 1 above.

Table 2: Projections of Double 3 configuration

Projection	Hilbert Function	Basis	Properties
$\mathbb{P}^3$	$1, 4, 10, 18, 24, \to 6d$	$3^2, 4^3$	
$\mathbb{P}^4$	$1, 5, 12, 18, 24, \to 6d$	$2^3, 3^2$	
$\mathbb{P}^5$	$1, 6, 12, 18, 24, \to 6d$	2^9	S

In the case of $\mathbb{P}^5$ numerical difficulties were encountered, it was necessary to add lines in an order to maximize the number of intersection points at each step.

4.3 The Double-four.

The double 4 configurations consists of lines 1,2,3,4,7,8,9,10 of Figure 1. A table for the 12×16 $M(\theta)$ as in Figure 3 has too many parameters. From [7, Th. 2.7] it follows that in addition to row operations on the pair x_i, y_i of columns, one can multiply an entire column or entire row by a non-zero constant. Given a line ℓ_i so that the only lines intersecting it are ℓ_j where $i < j$. Then by a series of all these operations the third point on this line can be denoted by $(1, 1)$. So the abstract seminormal $S_{(1)}$ for the double-4 is determined by a matrix $M(\theta)$ with only 8 parameters. In [7] we actually used even fewer, but this satisfactory for us. The first 4 lines have no parameters, all the parameters are in the lower right 4×8 matrix which looks like

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & b_7 & b_8 & \\ b_1 & b_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_3 & b_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_5 & b_6 & 0 & 0 \end{bmatrix} \tag{7}$$

I will do 3 experiments. For the first I simply replace all parameters by random real numbers $-1 < r < 1$. In this case $M(\theta)$ has rank 12 so $S_{(1)}$ projects directly into $\mathbb{P}^3$.

Motivated by the discussion of the double-four in [7] I apply the exact fraction free Gaussian Elimination to the unevaluated $M(\theta)$. The result is that all entries in the bottom row are polynomials in the parameters $b_1, \ldots, b_8$ and setting each entry to zero gives the simple simultaneous solution

$$b_4 = b_2,\ b_5 = -b_3,\ b_6 = -b_1,\ b_7 = -\frac{b_3}{b_2},\ b_8 = -\frac{b_1}{b_2} \tag{8}$$

Thus any non-zero random evaluation gives $M(\theta)$ of rank 11. In fact, for the random values I use, the smallest singular value was $0.4 * 10^{-15}$ while the next smallest was about 0.87 so the numerical rank was quite definitively 11. This $S_{(1)}$ projects directly into $\mathbb{P}^4$ and also into $\mathbb{P}^3$, giving the second two experiments.

The results are given in Table 3.

Table 3: Projections of Double 4 configuration

Experiment	Projection	Hilbert Function	Basis	Properties
1	$\mathbb{P}^3$	$1, 4, 10, 20, 28, \rightarrow 8d - 4$	4^7	
2	$\mathbb{P}^4$	$1, 5, 12, 20, 28, \rightarrow 8d - 4$	2^3	S,CI
3	$\mathbb{P}^3$	$1, 4, 10, 19, 28, \rightarrow 8d - 4$	$3, 4^3$	

The double 4 in experiment 2 is seminormal by [6, Ex. 4.4], seminormality does not follow from results of this paper since $HP_S(1) \neq H_S(1)$ and we only directly know the degree 1 part of S. This is a complete intersection. In fact, using a non-generic choice of parameters, in this case $b_1 = b_2 = b_3 = 1$ one also gets a seminormal complete intersection with the same Hilbert function. A Gröbner basis method gives the very pleasant homogeneous system

$$\begin{cases} xu - xy = 0 \\ xy - yz - yu + yv = 0 \\ zv = 0 \end{cases} \tag{9}$$

Note that two distinct double-fours are obtained in $\mathbb{P}^3$, one with seminormalization satisfying $HF_S(1) = 4$ and not contained in a cubic whereas the other has the generic example in $\mathbb{P}^4$ as its seminormalization and does lie in a cubic.

5 Classical Constructions related to Schläfli's Double-Six

In H. S. M. Coxeter's review [2] of Volume II of Ludwig Schläfli's collected works he says that one paper

> ... is modestly entitled "An attempt to determine the 27 lines upon a surface of the third order, and to divide such surfaces into species in reference to the reality of the lines upon the surface." The existence of 27 such lines had already been discovered by Cayley and Salmon, but this paper of 1856 gives the first complete discription of this configuration ...

This section is motivated by the discussion of Schläfli's result in Hilbert and Cohn-Vossen [11]. I will work exclusively in real projective 3-space in this section.

Start with an experiment, take three random lines in $\mathbb{P}^3$, or computationally, three 4×2 real matrices with random entries. Check that these lines are skew. Using Algorithm 3 to find the ideal of the union of these three lines one gets Hilbert function and polynomial $1, 4, 9, 12, \rightarrow\ 3t + 3$ and a basis of one quadratic and 4 cubics. So from either the Hilbert function or basis we see that there is a unique quadratic containing these three skew lines. Such a projective quadratic is called a hyperboloid and contains not just one, but two rulings of mutually skew lines [11, p.15].

The main new computational tool used in this section finds the 2 lines through a given point on a hyperboloid. The idea is that a line contained in a surface is tangent to the surface at each point of the line. Since we can obtain the equation $h = 0$ of the surface from Algorithm 3 we can easily obtain the equation of the tangent plane at each point P, i.e.

$$T = \frac{\partial h}{\partial x}\Big|_P x \ + \ \frac{\partial h}{\partial y}\Big|_P y \ + \ \frac{\partial h}{\partial z}\Big|_P z \ + \ \frac{\partial h}{\partial w}\Big|_P w \ = 0 \tag{10}$$

where I am using x, y, z, w as my projective coordinates. Thus we need to find a component of the common solution to $h = 0$ and equation (10). Since we are looking for a line and we already know one point it is enough to find one other *witness point* [19] on each line. To do this augment the two equations with two more, a random homogeneous linear equation and an equation of the form $ax + by + cz + dw - 1 = 0$ to keep away from $(0, 0, 0, 0)$. One should get two solutions Q_1, Q_2. The two lines are then the lines spanned by P, Q_1 and P, Q_2.

Summarizing

Algorithm 4: Find lines through P on hyperboloid $h = 0$
Given: Point P and hyperboloid $h = 0$

- Construct tangent plane T at P to surface $h = 0$.
- Let f, g be random homogeneous linear functions
- Solve system $[h, T, f, g - 1]^\top$ for two solutions Q_1, Q_2.

Output: Lines spanned by P, Q_1, and P, Q_2.

It is necessary to have a good numeric solver available, I have used an external call to PHCpack [22]. I also find it useful to check the results for accuracy by testing a random point on the output lines to make sure it lies very near $h = 0$. Sometimes it is necessary to re-run the algorithm to get the desired accuracy.

5.1 Unions of Lines in hyperboloidal Position

I illustrate the use of Algorithm 4 by calculating the ideals of numerically generic $p \times q$ configuration of p lines from one ruling on a hyperboloid and q lines from the other, see [5, Ex. 7]. When all lines lie in a single hyperboloid Hilbert and Cohn-Vossen say these lines are in [11, p. 164] *hyperboloidal position.*

When $p, q < 3$ the constructions follow by random selection of the lines and/or the methods of §4. If $p \geq 3$ then start with 3 random, hence skew, lines and find the hyperboloid $h = 0$ using Algorithm 3 as above. Then by picking q random points on one of these lines as intersection points one can calculate the q lines in the other ruling by Algorithm 4. If necessary, picking random points on one of these lines then Algorithm 4 will give the additional lines in the first ruling.

For my experiment I began by generating 3 random lines in $\mathbb{P}^3$ and used Algorithm 3 to find the unique hyperboloid containing them. Then I randomly picked the 5 intersection points on one of the lines. From each of these points I used Algorithm 5 to produce a line in the other ruling. I finally picked two random points on the first of these lines in the second ruling and then used Algorithm 5 to find the final two lines for the first ruling. I checked that each line in the first ruling did intersect each line in the second and also the the five lines in each ruling were, in fact, skew. The data from my experiment is given in Table 4, and the patterns are more striking given the randomness of the construction.

As before "S" stands for seminormal while "CI" means complete intersection. The determination of seminormality comes from [5], note however as in [6, §2] it follows from Theorem B that if $HF_A(d) < HP(d)$ for any $d \geq 1$ then the configuration cannot be seminormal. In particular this applies to the $4 \times q$ configuration for $q \leq 2$. The determination of complete intersection is based on explicit calculation of ideal generators.

As I reached the end of the chart the calculations became more difficult, with numerical issues arising. The construction based on Algorithm 5 is not as accurate as the projections of the last section.

Table 4: The $p \times q$ configuration in hyperboloidal position.

p	q	Hilbert Function	Basis	Properties
2	0	$1, 4, 6, 8, 10, \to 2d + 2$	2^4	S
2	1	$1, 4, 7, 10, 13, \to 3d + 1$	2^3	S
2	2	$1, 4, 8, 12, 16, \to 4d$	2^2	S,CI
3	0	$1, 4, 9, 12, 15, \to 3d + 3$	$2, 3^4$	
3	1	$1, 4, 9, 13, 17, \to 4d + 1$	$2, 3^3$	
3	2	$1, 4, 9, 14, 19, \to 5d - 1$	$2, 3^2$	S
3	3	$1, 4, 9, 15, 21, \to 6d - 3$	$2, 3$	S,CI
4	0	$1, 4, 9, 16, 20, \to 4d + 4$	$2, 4^5$	
4	1	$1, 4, 9, 16, 21, \to 5d + 1$	$2, 4^4$	
4	2	$1, 4, 9, 16, 22, \to 6d - 2$	$2, 4^3$	
4	3	$1, 4, 9, 16, 23, \to 7d - 5$	$2, 4^2$	S
4	4	$1, 4, 9, 16, 24, \to 8d - 8$	$2, 4$	S,CI
5	4	$1, 4, 9, 16, 25, 34, \to 9d - 11$	$2, 5^2$	S
5	5	$1, 4, 9, 16, 25, 35, \to 10d - 15$	$2, 5$	S,CI

5.2 The Double-six

I will closely follow the construction given on pages 164–167 of Hilbert and Cohn-Vossen [11]. They start with the following assertion about lines in $\mathbb{P}^3$.

Given three skew lines a, b, c and a fourth line d that is not in, or tangent to, the hyperboloid determined by a, b, c there are exactly two lines ℓ_1, ℓ_2 which intersect all four of a, b, c, d.

Existence is given by a simple construction. Since the hyperboloid is a surface of degree 2 and d is neither in, nor tangent to, the surface d must meet the hyperboloid in two distinct points. Since the hyperboloid is a doubly ruled surface through each of these points there is a line in the opposite ruling to a, b, c through that point. These are the desired lines, they must be distinct since otherwise d would meet the common line in two distinct points and thus be the common line. But d is not in the hyperboloid.

To see that there are no other lines note that any line ℓ meeting all of skew lines a, b, c must lie in the hyperboloid because it must meet the hyperboloid in at least three distinct points. But then ℓ must meet d at the points where d meets the hyperboloid so ℓ is one of the lines constructed above.

Computationally, I can carry out this construction using Algorithm 3 to explicitly find the hyperboloid, then use a numeric solver to find the two points where d intersects this hyperboloid and then find the desired lines from Algorithm 4. Note that, rarely, even when a, b, c, d are all real lines, the soluton lines ℓ_1, ℓ_2 may be imaginary. Below a, b, c, d and ℓ_1 will be known and the problem will be to find ℓ_2. There are the two points where d intersects the hyperbolid and two ruled lines through each of these. Only one of the four choices gives ℓ_2, each choice must be tested until the correct choice is found.

One other simple construction will be needed. Given a line ℓ I will need to find a random line meeting ℓ. Computationally ℓ will be represented by a 4×2 real matrix $\mathbf{n}$. I must get a second 4 matrix $\mathbf{m}$ as random as posible but with $\text{rank}[\mathbf{n}|\mathbf{m}] = 3$. So I choose one column of $\mathbf{m}$ to be a random linear combination of the columns of $\mathbf{n}$ and the second to be random. This will give me the desired line with probability one.

Table 5 describes my construction following that of Hilbert and Cohn-Vossen. The line number at the left refers to Figure 1, and lines are added in the order of the construction in [11]. At each step I calculate the Hilbert function and degree of basis elements of the ideal of the union of lines constructed so far, basis notation follows the previous tables.

Table 5: Construction of the Double-six

line	construction	Hilbert function	basis
1	random line	$1, 2, 3, \to d+1$	1^2
8	random line meeting line 1	$1, 3, 5, \to 2d+1$	$1, 2$
9	random line meeting line 1	$1, 4, 7, \to 3d+1$	2^3
10	random line meeting line 1	$1, 4, 9, 13, \to 4d+1$	$2, 3^3$
11	random line meeting line 1	$1, 4, 10, 16, 21, \to 5d+1$	$3^3, 4$
6	other line meeting 8,9,10,11	$1, 4, 10, 16, 22, \to 6d-2$	3^3
12	random line meeting line 1	$1, 4, 10, 19, 26, \to 7d-2$	$3, 4^5$
5	other line meeting 8,9,10,12	$1, 4, 10, 19, 27, \to 8d-5$	$3, 4^4$
4	other line meeting 8,9,11,12	$1, 4, 10, 19, 28, \to 9d-8$	$3, 4^3$
3	other line meeting 8,10,11,12	$1, 4, 10, 19, 29, \to 10d-11$	$3, 4^2$
2	other line meeting 9,10,11,12	$1, 4, 10, 19, 30, 41, \to 11d-14$	$3, 4, 5$
7	other line meeting 2,3,4,5	$1, 4, 10, 19, 30, 42, \to 12d-18$	$3, 4$

The construction does not specifically require that line 7 meet line 1, however Hilbert tells us that it will, in theory. Thus a required check on the construction is to test whether line 7 does in fact meet line 1. I find that it does with tolerance 10^{-8}, i.e. if $\mathbf{n}_1, \mathbf{n}_7$ are the matrices for lines 1 and 7 then by calculation the smallest singular value of $[\mathbf{n}_1|\mathbf{n}_7]$ is less than 10^{-8}.

From either the Hilbert function or Algorithm 3 this computation also shows that this double-six is contained in a cubic, as also expected by the theory. It also shows that the double-six is a complete intersection of the cubic and a quartic, a fact that I have not seen explicitly mentioned. The double-six is seminormal by [5].

5.3 The Double-five

For completeness I mention the double-five. Lines 1–5, 7–11 of the double-six above form a double-five as a subvariety of the double-six. The Hilbert function is $1, 4, 10, 19, 30, 40, \to 10t-10$. The basis for the ideal is $3, 4, 5^2$ in the notation of this paper so this is not a complete intersection. In fact by Theorem B or [6, §2] since $HF_A(3) < HP(3)$ this double-five in a cubic cannot be seminormal. A double-five not in a cubic can be constructed in a similar method to the double-six. Here start with both lines 1 and 7 as random lines, lines 8,9,10,11 are random lines meeting line 1 and lines 2–5 are constructed by the hyperboloid method of the last subsection. The resulting Hilbert function is $1, 4, 10, 20, \rightarrow 10t - 10$ and the basis consists of 5 fourth degree forms, i.e. 4^5. This is seminormal by [5] but, like the non-seminormal double-five, not a complete intersection.

One interesting observation we made in [5] was that if any line is removed from either double-five above then $HP = 9t - 7$ so $HF_A(2) = 10 < 11 = HP(2)$ hence the resulting union is not seminormal by Theorem B. In particular the double-five not in the cubic is a seminormal union of 10 lines such that any 9-line subvariety is not seminormal.

5.4 Bertini, The Double-two and Schläfli again

As additional verification of my results I attempted to run the calculated ideals on the numerical algebraic geometry software Bertini [1].

Bertini was able to run the complete intersection ideals as homogeneous sysystems but I was not able to get results on those homogeneous ideals of realizations in $\mathbb{P}^n$ when the number of generators exceeded $n-1$. However running Bertini on subsets of $n-1$ generators was useful as the numerical primary decomposition of these complete intersection systems contain the lines of the configurations together with one or more additional one-dimensional components. I was able to get the desired results in most of my experiments by relaxing the Bertini default tolerances slightly.

In a few cases the extra components were of degree 1, i.e. were additional lines. Often this had an immediate explaination, for example in the $p \times q$ configurations in hyperboloidal position if $q < p$ then the complete intersection of the quadratic and one other generator was the $p \times p$ configuration, see Table 4. The surprise was double-three in $\mathbb{P}^3$ using the two cubic generators. The resulting complete intersection was a *triple-three* configuration consisting of 3 families of 3 skew lines such that each line from a given family met exactly two lines of each of the other families.

This can actually be understood from the double 2 configuration, for example lines 1,2,7,8 of Figure 1. The double-two configuration generically realized in $\mathbb{P}^3$ has Hilbert function $1, 4, 9, 14, \rightarrow 4d + 2$ and the ideal is generated by a quadratic, two cubics and a biquadratic. Intersecting lines 1,8 line in a common plane the equation of which is easily calculated as the linear generator found by Algorithm 3. A Bertini experiment shows that the intersection of this linear equation with one of the cubic generators of the ideal consists of three lines. The new line (not 1,8) can be calculated by another run of Bertini choosing two witness points on the appropriate component, which is found by trial and error. This line meets all 4 lines of the double 2 but does not, since the construction is generic, go through either of the original intersection points. It is easily seen that this line must be the intersection of the plane containing lines 1,8 and the plane containing 2,7, so starting with 2,7 would produce the same line.

This explains the triple-three found as the intersection of two cubics containing the double-three in $\mathbb{P}^3$. The double-three contains 3 double-two configurations, each of which produces, inside of any cubic containing the whole configuration, a third line. So the intersection of any two cubics containing the double-three must contain 9 lines.

This was Schläfli's trick in producing the 27 lines on the cubic surface. Starting with a double-six he noted that he had 15 double-twos inside, each producing an additional distinct line inside the original cubic. Thus one method for constructing the 27 lines on a generic cubic would be to construct a generic double-six as in Table 5 and use the method above with Bertini to obtain the other 15 lines. A different approach will be sketched in the Appendix.

6 Conclusion

Most of the results given in this paper were previously known and those that may be new are incidental to the main purpose of this paper, which is to demonstrate the effectiveness of Algorithms 1–3. These algorithms worked well most of the time, for a few of the calculations the algorithms needed to be run several times or in some cases a different order in which the lines was added gave better results. Finding the monomial basis in Algorithm 2 and the reduction step in Algorithm 3 were most likely to numerically unstable for problems with many generators of higher degrees. When there was a problem typically this was identified by failure to achieve consistent results between Algorithm 1, Algorithm 2 and the known Hilbert polynomial. Thus Algorithm 2, which was intended to insure a high enough degree is used in Algorithm 3, also served as a useful check on the numerics. Running time was not an issue for the size of problems considered, however developing a faster method of constructing the Macaulay Matrix would be helpful.

Most of all this paper was an enjoyable endeavor, connecting classical constructions with modern numerical methods. I thank Andrew Sommese and the organizers of this conference for indirectly motivating this work.

7 Appendix: Further details on the Double-six

For those readers who would like more details I give the ideal $\langle f, g\rangle$ of the double-six of Section 5.2 in Figure 5 along with 12 of the intersection points, 2 on each line in Figure 6. The double-six can be reconstructed with this information. Since I worked with a tolerance of 10^{-8} I give this data to only 10 decimal places. The reader should find that the residues of these points under f, g, or random linear combinations of two of these points on the same line, are less than 10^{-8} while a comparable random point not on the double-six will generally have residue larger than 10^{-2}.

To find the equation a specific line, say ℓ_1 produce the matrix with columns the two points on $\ell_1, \ell_1 \cap \ell_{10}$ and $\ell_1 \cap \ell_{11}$ with approxi-left null-space columns of N

$$M = \begin{bmatrix} -.5854379199 & -.6479860778 \\ .2645298328 & .3232470967 \\ .1598710137 & .1808024092 \\ .7494849355 & .6655342563 \end{bmatrix} \qquad N = \begin{bmatrix} .19285566669 & .5634975329 \\ -.1468158563 & .8121072532 \\ .9701709632 & .01157912904 \\ -.00448310687 & .15105715298 \end{bmatrix}$$

$$
\begin{aligned}
f =& .03351127663x^3 - .1429711953yx^2 + .07241841360zx^2 + .1081936936wx^2 - .4593197466y^2x \\
&+ .1709914640zyx + .2727231459wyx + .01234636776z^2x - .02181828213wzx - .1479770049w^2x \\
&- .2842041187y^3 + .1210633430zy^2 + .1834877873wy^2 + .08489035820z^2y + .04705532012wzy \\
&- .00016769334w^2y - .03673636650z^3 - .06650068095wz^2 - .04092257041w^2z - .00919882902w^3 \\
g =& .1774520591x^4 - .9232579496w^3y - .0070359194w^4 + .1443598713y^4 - .001882858627w^2z^2 \\
&+ .1182279125w^2y^2 + .4734759417wz^3 + .4530059073yx^3 - .1637446827y^3z - .08035955279z^3 * y \\
&+ .1448736933z^2y^2 + .01738014638z^4 - .03007426727w^3z - .2429350862w^3x - .01343420698zx^3 \\
&+ .2228641957wx^3 + .2802504294y^2x^2 - .1648570426z^2x^2 + .1017112211y^3x + .03458059387z^3x \\
&+ .5953944404w^2x^2 - .1474904269wy^3 + .05969748448wzyx + .1556125457wy^2x + .1694866052w^2yx \\
&- .1011637448wz^2x - .9899487214w^2zx - .1981908659wz^2y - .08409866175w^2zy + .9728937572y^2xz \\
&+ .4806949231wyx^2 - .6492161240wzx^2 + .2542457775wzy^2 - .1584889276z^2xy + .1422633184zyx^2
\end{aligned}
$$

Figure 5: Equations of Double-six of §5.2

$$
\ell_2 \cap \ell_7 = \begin{bmatrix} .4037409402 \\ -.3365762254, \\ -.3819703921, \\ .7601370385 \end{bmatrix} \qquad
\ell_6 \cap \ell_7 = \begin{bmatrix} .1962272447 \\ .1415703611, \\ .8698650449 \\ -.4298691720 \end{bmatrix} \qquad
\ell_3 \cap \ell_8 = \begin{bmatrix} -.1124878417 \\ -.1808569060, \\ -.8420630296 \\ .4955472925 \end{bmatrix}
$$

$$
\ell_6 \cap \ell_8 = \begin{bmatrix} -.1986275803 \\ -.1225250017, \\ -.8590046133 \\ .4556816680 \end{bmatrix} \qquad
\ell_2 \cap \ell_9 = \begin{bmatrix} .5545298037 \\ -.4116396051 \\ -.4236036036 \\ .5861821554 \end{bmatrix} \qquad
\ell_5 \cap \ell_9 = \begin{bmatrix} -.4312661269 \\ .3659746522 \\ -.4741137979 \\ .6747504638 \end{bmatrix}
$$

$$
\ell_1 \cap \ell_{10} = \begin{bmatrix} -.5854379199 \\ .2645298328 \\ .1598710137 \\ .7494849355 \end{bmatrix} \qquad
\ell_5 \cap \ell_{10} = \begin{bmatrix} .2256651716 \\ -.3311382640 \\ .2945679289 \\ -.8675554251 \end{bmatrix} \qquad
\ell_1 \cap \ell_{11} = \begin{bmatrix} -.6479860778 \\ .3232470967 \\ .1808024092 \\ .6655342563 \end{bmatrix}
$$

$$
\ell_4 \cap \ell_{11} = \begin{bmatrix} -.1032272442 \\ -.4277069904 \\ -.8949543552 \\ -.07394300895 \end{bmatrix} \qquad
\ell_3 \cap \ell_{12} = \begin{bmatrix} -.4556881447 \\ .2836272231 \\ -.07622228238 \\ .8402940418 \end{bmatrix} \qquad
\ell_4 \cap \ell_{12} = \begin{bmatrix} -.2937259451 \\ .3532304135 \\ -.3929967974 \\ .7965593898 \end{bmatrix}
$$

Figure 6: Selected points on the Double-six (homogeneous coordinates in $\mathbb{P}^3$)

Then the desired generators for the ideal of ℓ_1 are

$$\ell_1 = \begin{cases} .1928556667x - .1468158563y + .97017096324z - .004483106866w \\ .563497532x + .8121072532y + .0115791290z + .1510571530w \end{cases}$$

One can also find the plane containing, say, ℓ_1, ℓ_8 by adding the two points $\ell_3 \cap \ell_8$ and $\ell_6 \cap \ell_8$ to matrix M above and obtaining the equation from the single vector spanning the approxi-nullspace of this. The system containing this equation and the like equation obtained from lines ℓ_2, ℓ_7 then is an equation for the third line in any cubic containing the double-two of lines 1,2,7,8.

$$\ell_{13} = \begin{cases} -.37598221088x - .8932506730y + .2462544645z + .0099674946w \\ .64852803510x - .1736828755y + .333375533z + .6618960642w \end{cases}$$

In this manner one can construct the equations of all 27 lines in the cubic surface defined by f above.

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