Numerical Local Rings and Local Solution of Nonlinear Systems

Barry H. Dayton
Department of Mathematics,
Northeastern Illinois University,
Chicago, IL 60625
bhdayton@neiu.edu

January 18, 2007

Abstract

Adapting a 1915 method of Macaulay, a calculation of the local ring of an isolated zero of a polynomial system $\{f_1, f_2, \ldots, f_t\} \subseteq \mathbb{C}[x_1, x_2, \ldots, x_s]$ is given using floating point arithmetic. This calculation gives the local ring as a quotient ring of $\mathbb{C}[x_1, x_2, \ldots, x_s]$ or by a representation as a matrix algebra. A key tool is an approximate reverse reduced row echelon form algorithm. By relaxing the tolerance one gets information on zeros in a small Euclidean neighborhood of the given zero. This technique, which may be useful in the endgame stage of the homotopy continuation method, is applied to analytic as well as polynomial systems.

1 Introduction

In 1915 Macaulay [12] introduced dialytic and inverse arrays which together implicitly described the structure of the local ring in the case of an isolated solution of a system of polynomial equations. For example, Macaulay's arrays are reproduced by T. Mora [14]. Recently, Zhonggang Zeng and the author [5] used similar arrays to find local

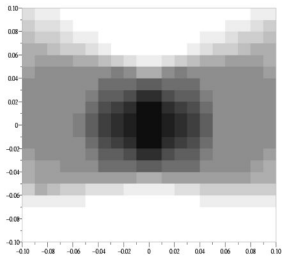

Figure 1: Critical Neighborhood

information about an isolated zero given an approximate location of the zero and/or approximate equations. We called our arrays the *multiplicity* and *dual* matrices.

In particular, in [5] the multiplicity of the zero was shown to be the approxi-dimension of the dual space. Due to space limitations we were not able to include the graphics in Figure 1 which show the calculated multiplicity for various approximations of the zero. For this picture the multiplicity matrix and its column rank deficiency were calculated for approximately 400 values of an approximate zero near an exact real zero of a two dimensional system. The darker colors represent higher multiplicity so that calculating the Multiplicity matrices at a point in the darkest neighborhood around the exact zero gives the correct multiplicity and other local information. In the next darker neighborhood, if the approximation to the zero is located here, the method indicates a multiple point but with lower multiplicity. This continues until the white area, where the point used to calculate the multiplicity matrices is determined to not be an approximate zero. The size of these critical neighborhoods depends on the tolerance; with a loose tolerance they were large, but with a tight tolerance they were smaller.

Thus our method worked reasonably well when the zero was well isolated, i.e. there was a large neighborhood of the point with only this one zero. However, it appeared that an annoying problem with our method arose when there were distinct but close isolated zeros. In this case, for some choices of tolerance, an approximate zero might lie in the critical neighborhood of two or more of these points. In this case we would get strange, and often unexplained, large multiplitiicy values.

This paper looks at this method in more detail, examining the structure of the local ring of a multiple, or appproximate multiple zero. When this approximate zero is, or is near, a cluster of zeros the method may be able to seperate out these zeros.

Thus we obtain a *numerical local ring*, or NLR. We can calculate a non-reduced Gröbner basis with respect to a global degree ordering for this NLR. Alternatively we give a representation for this local ring as a matrix algebra. In either case an important tool is the *approximate reverse reduced row echelon form*, ARRREF. A similar method was also used recently by Robin Scott in her Masters thesis under Greg Reid [15]. In the exact case all eigenvalues of matrices in the maximal ideal vanish. However, in the numerical case some of these eigenvalues may not be zero and one can apply the Möller-Stetter method [13, 17] to calculate potential additional zeros of the original system. By deliberately loosening the tolerance in the ARRREF one may enlarge the critical region and discover additonal nearby points.

We give several numerical examples, including an example of an analyic system of Griewank and Osborne [6] where this NLR method finds 4 nearby complex zeros near the origin. We explain how this method may provide a useful tool in the endgame stage for the homotopy continuation method [2, 16, 18].

The local finite dimensionality is necessary for this method, but not global finite $\mathbb{C}$-dimension. As long as we stay reasonably far from higher dimensional components of the solution set, we generally get useful results from our method.

2 The Local Array

Z. Zeng and the author describe the multiplicity matrix in [5]. In this paper the term *Macaulay's local array* or simply *local array* will be used. Other authors have used other names, for example Leykin, Verschelde and Zhao use the term *generalized Jacobian* [9].

For the reader's convenience we will summarize that construction here. Let $f_1, f_2, \ldots, f_t$ be a list of t equations in s variables $x_1, x_2, \ldots, x_s$. To get isolated solutions it is necessary for $t \geq s$. By *equation* is meant a polynomial or analytic function, that is a function with partial derivatives of all orders and that are, at least, locally approximated by polynomials. For a non-negative integer array $\mathbf{j} = [j_1, \ldots, j_s]$ write $|\mathbf{j}| = j_1 + j_2 + \cdots + j_s$ and $\mathbf{x}^{\mathbf{j}} = x_1^{j_1} x_2^{j_2} \ldots x_s^{j_s}$. So $\mathbf{x}^{\mathbf{j}}$ has total degree $|\mathbf{j}|$. Assume a global degree ordering is given on the monomials. As in [5] we use the differentiation operator

$$\partial_{\mathbf{x}^{\mathbf{j}}} \equiv \partial_{x_1^{j_1} \cdots x_s^{j_s}} \equiv \frac{1}{j_1! \cdots j_s!} \frac{\partial^{j_1+\cdots+j_s}}{\partial x_1^{j_1} \cdots \partial x_s^{j_s}}.$$

Let $\hat{\mathbf{x}}$ be a point in $\mathbb{C}^s$, write $\partial_{\mathbf{x}^j}(g)(\hat{\mathbf{x}})$ to indicate taking the partial derivative of function g and evaluating at $\hat{\mathbf{x}}$. Note that $\partial_1(g)(\hat{\mathbf{x}}) = \partial_{\mathbf{x}^{\mathbf{0}}}(g)(\hat{\mathbf{x}})$ is simply evaluation of g at $\hat{\mathbf{x}}$, i.e. $g(\hat{\mathbf{x}})$.

The *local array of degree k at $\hat{\mathbf{x}}$*, $\mathbf{L}(k, \hat{\mathbf{x}})$ is the $t\binom{k+s-1}{s} \times \binom{k+s}{s}$ matrix with columns indexed by the differentials $\partial_{\mathbf{x}^\mathbf{j}}$ for $\mathbf{j} \leq k$ or, more commonly, just by the $\mathbf{x}^\mathbf{j}$ where the columns indexed by $\mathbf{x}^\mathbf{j}$ of higher total degree are to the right of those with lower degree. In particular, the left hand column has index 1 for the evaluation functional. The rows will be indexed by the functions $\mathbf{x}^\mathbf{i} f_\alpha$ for $|\mathbf{i}| < k$, $\alpha = 1, \ldots, t$. Again these will be grouped by $|\mathbf{i}|$ with smaller total degree having lower row index . In particular the first t rows are indexed by $f_1, \ldots, f_t$. Within a degree the grouping method does not matter.

The entry in the row indexed by $\mathbf{x}^\mathbf{i} f_\alpha$ and column indexed by $\mathbf{x}^\mathbf{j}$ is $\partial_{\mathbf{x}^\mathbf{j}}((\mathbf{x}^\mathbf{i} - \hat{\mathbf{x}}^\mathbf{i}) f_\alpha)(\hat{\mathbf{x}})$

By the co-rank of a matrix L we mean the dimension of the nullspace of L, or the dimension of the approxi-nullspace of L if we are using numerical linear algebra. This is defined in [11] or one may obtain reasonably similar results by counting the number of columns minus the number of singular values of L greater than the tolerance ε. Using this formulation then $\varepsilon = 0$ gives the exact case.

The reason for using $\partial_{\mathbf{x}^\mathbf{j}}((\mathbf{x}^\mathbf{i} - \hat{\mathbf{x}}^\mathbf{i}) f_\alpha)(\hat{\mathbf{x}})$ instead of simply $\partial_{\mathbf{x}^\mathbf{j}}(\mathbf{x}^\mathbf{i} f_\alpha)(\hat{\mathbf{x}})$ in the definition of $\mathbf{L}(k, \hat{\mathbf{x}})$, which would still give a row-equivalent matrix, is that one obtains a block structure

$$\mathbf{L}(k, \hat{\mathbf{x}}) = \left[\begin{array}{c|c} \mathbf{L}(k-1, \hat{\mathbf{x}}) & A \\ \hline \mathbf{0} & B \end{array}\right] \tag{1}$$

where A, B have $\binom{k+s-1}{k}$ columns. It is then easily seen that co-rank$\mathbf{L}(k-1, \hat{\mathbf{x}}) \leq$ co-rank $\mathbf{L}(k, \hat{\mathbf{x}})$. Thus we define $H_\varepsilon(k, \hat{\mathbf{x}}) =$ co-rank$\mathbf{L}(k, \hat{\mathbf{x}}) -$ co-rank$\mathbf{L}(k, \hat{\mathbf{x}})$

In the exact polynomial case, i.e. we are using polynomials with exact coefficients and exact $\hat{\mathbf{x}}$, then we recover the *Hilbert function* $H_0(k, \hat{\mathbf{x}})$ of the local ring. This sequence is very well behaved as discussed in [5] and expanded upon in the next section. In the numerical case the sequence $\{H_\varepsilon(k, \hat{\mathbf{x}})\}$ may be less well behaved.

In this paper we will need a *stable criterion*. This will mean that the matrix $\left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right]$ in (1) have independent columns, or equivalently that the last $\binom{k+s-1}{k}$ columns of $\mathbf{L}(k, \hat{\mathbf{x}})$ are independent. The following is clear.

Lemma 1 *Consider the following conditions where A, B are as in* (1).

i) *B is of full column rank.*
ii) *$H_0(k, \hat{\mathbf{x}}) = 0$.*

iii) $\mathbf{L}(k, \hat{\mathbf{x}})$ *satisfies the stable condition.*

Then i) *implies* ii) *implies* iii).

As is well known, once the Hilbert function of a finite dimensional local $\mathbb{C}$-algebra takes the value 0, then it remains zero. At this point the cummulative Hilbert function $H_0, H_0+H_1, H_0+H_1+H_2, \ldots$ *stabilizes* at the dimension of the algebra. Thus the stable criterion is satisfied when the cumulative Hilbert function stabilizes.

3 Representation of Local Rings in the Exact Case

As mentioned in the introduction, Macaulay's local arrays give a computational way to get information normally associated with the local ring. It is a curious fact, to our knowledge so far unreported, that, in theory, one can use local arrays to computationally get a full description of this local ring by a presentation or by concrete representation of the local ring of an isolated zero as a matrix algebra.

Consistent with the rest of this paper we work over $\mathbb{C}$, however the results of this section should work over any algebraically closed field, with the understanding that $\hat{\mathbf{x}}$ is an isolated zero means the local ring is a finite dimensional algebra over that field. Suppose we have polynomial equations $f_1, \ldots, f_t \in \mathbb{C}[x_1, \ldots, x_s] = \mathbb{C}[\mathbf{x}]$. We write $A = \mathbb{C}[x_1, \ldots, x_s]/\langle f_1, \ldots, f_t \rangle$ where $\langle f_1, \ldots, f_t \rangle$ is the ideal of $\mathbb{C}[x_1, \ldots, x_s]$ generated by $f_1, \ldots, f_s$. We denote by $\mathcal{O}_{\hat{\mathbf{x}}}(A)$, or more simply by $\mathcal{O}_{\hat{\mathbf{x}}}$, the local ring at the zero $\hat{\mathbf{x}}$ of the system. By local ring at $\hat{\mathbf{x}}$ we mean, as in [7, A.8],

$$\mathcal{O}_{\hat{\mathbf{x}}} = A_{M_{\hat{\mathbf{x}}}} = \left\{ \frac{f}{g} \middle| f, g \in A, g(\hat{\mathbf{x}}) \neq 0 \right\}$$

where $M_{\hat{\mathbf{x}}}$ is the maximal ideal of A at $\hat{\mathbf{x}}$. If A itself is finite dimensional over $\mathbb{C}$ then the following is immediate from the isomorphism of A with the product of its local rings, eg. [4, Ch. 4,2.2].

Lemma 2 *Let $A = \mathbb{C}[\mathbf{x}]/\langle f_1, \ldots, f_t \rangle$ and suppose that the local ring $\mathcal{O}_{\hat{\mathbf{x}}} = A_{M_{\hat{\mathbf{x}}}}$ is finite dimensional as a $\mathbb{C}$-algebra, then the natural map $\mathbb{C}[\mathbf{x}] \to \mathcal{O}_{\hat{\mathbf{x}}}$ is a surjection.*

Proof: If $\mathbf{m}$ is the maximal ideal of $\mathcal{O}_{\hat{\mathbf{x}}} = \mathbf{m}^0$ then for any $\alpha \geq 0$

$$\dim_{\mathbb{C}} \mathcal{O}_{\hat{\mathbf{x}}} = \dim_{\mathbb{C}} \mathbf{m}^0 \geq \dim_{\mathbb{C}} \mathbf{m}^0 - \dim_{\mathbb{C}} \mathbf{m}^{\alpha+1} = \sum_{\beta=0}^{\alpha} \big(\dim_{\mathbb{C}} \mathbf{m}^{\beta} - \dim_{\mathbb{C}} \mathbf{m}^{\beta+1}\big) = \sum_{\beta=0}^{\alpha} \dim_{\mathbb{C}} \mathbf{m}^{\beta}/\mathbf{m}^{\beta+1}$$

Since the left hand side is finite, for some positive β we must have $\dim_{\mathbb{C}} \mathbf{m}^{\beta}/\mathbf{m}^{\beta+1} = 0$ By Nakayama's lemma $\mathbf{m}^{\beta} = 0$. Note that if β is the smallest such integer, then $\beta - 1$ is known in the literature as the *nil-index* of $\mathcal{O}_{\hat{\mathbf{x}}}$, in [5] we used the confusing term *depth*.

Now suppose $g \in A$ but not in $M_{\hat{\mathbf{x}}}$, then g is the image of $g \in \mathbb{C}[\mathbf{x}]$ so that $g(\hat{\mathbf{x}}) = r \neq 0, r \in \mathbb{C}$. Then

$$\frac{1}{g} = \frac{1}{r - (r - g)} = \frac{\frac{1}{r}}{1 - (1 - \frac{g}{r})} = \frac{1}{r}\big(1 + (1 - \tfrac{g}{r}) + \cdots + (1 - \tfrac{g}{r})^{\beta - 1}\big)$$

since $(1 - \frac{g}{r}) = \frac{1}{r}(r - g) \in \mathbf{m}$. So $\frac{1}{g}$ is in the image of $\mathbb{C}[\mathbf{x}]$. ∎

We now show how to compute the kernel $\mathbb{C}[\mathbf{x}] \to \mathcal{O}_{\hat{\mathbf{x}}}$. We can assume without loss of generality that $\hat{\mathbf{x}}$ is the origin $\mathbf{0}$.

Let $\mathbf{X} = [1, x_1, \ldots, x_s^k]$ be the vector of column indices for $\mathbf{L}(k, \hat{\mathbf{x}})$. Then I will view $F_k = \mathbf{L}(k, \hat{\mathbf{x}})\mathbf{X}^{\top}$ as a new system of equations. These equations are just truncations of the equations $\mathbf{x}^{\mathbf{i}} f_{\alpha}$ with the terms of degree greater than k set to zero. If $\hat{\mathbf{x}}$ were not an exact zero F_k might be inconsistent, but here $\hat{\mathbf{x}}$ is a zero of F_k. Note that if we had not assumed $\hat{\mathbf{x}} = \mathbf{0}$ then the equations would be Taylor expansions of $(\mathbf{x} - \hat{\mathbf{x}})^{\mathbf{i}} f_{\alpha}$ about $\hat{\mathbf{x}}$.

In the exact case, as mentioned in the last section, the stable criterion must hold for $\mathbf{L}(k, \hat{\mathbf{x}})$ when $k > d$ where d is the nil-index by [5, Theorem 1].

Lemma 3 *Let $\hat{\mathbf{x}} = \mathbf{0}$ be an exact isolated zero of the exact polynomial system $f_1, \ldots, f_t$ in $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \ldots, x_s]$. Let d be the nil-index for the local ring $\mathcal{O}_{\hat{\mathbf{x}}}$, then for $k > d$ $\mathcal{O}_{\hat{\mathbf{x}}} = \mathbb{C}[\mathbf{x}]/\mathcal{I}$ where $\mathcal{I} = \langle f | f$ is an entry of $F_k \rangle$, F_k as above.*

Proof: First of all since the nil-index is $d < k$ any monomial of degree k or greater vanishes in $\mathcal{O}_{\hat{\mathbf{x}}}$, therefore the entries of F_k which are of the form $x^{\mathbf{j}} f_i$ with all terms of degree greater than k truncated are elements of the kernel $\mathbb{C}[\mathbf{x}] \to \mathcal{O}_{\hat{\mathbf{x}}}$. Thus $\mathcal{O}_{\hat{\mathbf{x}}}$ is a subquotient of $\mathbb{C}[\mathbf{x}]/\mathcal{I}$ However by [5, Th. 2] the $\mathbb{C}$-dimension of $\mathcal{O}_{\hat{\mathbf{x}}}$ agrees with the $\mathbb{C}$-dimension of the $\mathbb{C}[\mathbf{x}]/\mathcal{I}$ and therefore $\mathcal{I}$ is all of the kernel $\mathbb{C}[\mathbf{x}] \to \mathcal{O}_{\hat{\mathbf{x}}}$. ∎

We will illustrate the ideas in this paper with the following example:

Example 1: Let $f_1 = 8x^4 - \frac{1}{7}x^3 + x - \frac{8}{7}y$, $f_2 = y^3 + x - \frac{8}{7}y$ and $F = \left[\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right]$. We consider the point $\hat{\mathbf{x}} = \mathbf{0} = (0,0)$. We find that $H_\varepsilon(k, \hat{\mathbf{x}})$ stabilzes at $k = 3$. Then

$$\mathbf{L}(3,\hat{\mathbf{x}}) = \begin{bmatrix}
0 & 1 & -\frac{8}{7} & 0 & 0 & 0 & -\frac{1}{7} & 0 & 0 & 0 \\
0 & 1 & -\frac{8}{7} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & -\frac{8}{7} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -\frac{8}{7} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -\frac{8}{7} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -\frac{8}{7} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{8}{7} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{8}{7} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{8}{7} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{8}{7} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{8}{7} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{8}{7}
\end{bmatrix}$$

There are 5 obvious duplicate rows so the rank of $\mathbf{L}(3,\hat{\mathbf{x}})$ is at most 7, and, in fact, it is 7. The equations generated by this matrix are f_1^t which is f_1 with the x^4 term truncated, f_2 and the equations $f_3 = x^2 - \frac{8}{7}xy$, $f_4 = xy - \frac{8}{7}y^2$. So the local ring is $\mathbb{C}[x,y]/\langle f_1^t, f_2, f_3, f_4\rangle$. This is not very illuminating, for example it is not obvious that $x^4 = 0$.

A better way to present this local ring is to apply the reverse reduced row echelon form algorithm (RREF) to this matrix, that is apply RREF starting from the bottom right corner, Alternatively, we can physically rotate $\mathbf{L}(3,\hat{\mathbf{x}})$ 180°, apply the usual RREF algorithm, and then rotate back. Eliminating zero rows we obtain

$$\mathbf{R} = \begin{bmatrix}
0 & -\frac{7}{8} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{7}{8} & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{49}{64} & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

which must happen since this local ring is a quotient of $\mathbb{C}[x,y]/\langle f_1, f_2\rangle$. This gives the much simpler presentation $\mathcal{O}_{\hat{\mathbf{x}}} = \mathbb{C}[x,y]/\mathcal{I}$ for $\mathcal{I} = \langle -\frac{7}{8}x + y, x^3\rangle$ as the other equations follow easily from these. For instance, we see that not only is $x^4 = 0$ but already $x^3 = 0$. It is moreover clear that $-\frac{7}{8}x + y, x^3$ is a Gröbner basis for the ideal $\mathcal{I}$. ∎

This example suggests the following

Theorem 1 *Assume* $\hat{\mathbf{x}} = \mathbf{0}$ *is an isolated zero of the system* $f_1, \ldots, f_s$ *in* $\mathbb{C}[\mathbf{x}]$. *Let* $H_0(k, \hat{\mathbf{x}}) = 0$ *and* R *be the reverse reduced row echelon form of* $\mathbf{L}(k, \hat{\mathbf{x}})$, X *be the row vector of column indices for* $\mathbf{L}(k, \hat{\mathbf{x}})$, $\mathcal{I}$ *the ideal generated by the entries of* $F_k = \mathbf{L}(k, \hat{\mathbf{x}})X^\top$ *so that (by Lemma 3)* $\mathcal{O}_{\hat{\mathbf{x}}} = \mathbb{C}[\mathbf{x}]/\mathcal{I}$. *Let* R *be the reverse reduced row echelon form of* $\mathbf{L}(k, \hat{\mathbf{x}})$ *and* $G_k = RX^\top$. *The entries of* G_k *form a Gröbner basis for* $\mathcal{I}$ *with respect to a global degree ordering compatible with* X.

Proof: The fact that $H_0(k, \hat{\mathbf{x}}) = 0$ implies both that k is larger than the nil-index of $\mathcal{O}_{\hat{\mathbf{x}}}$, by [5], and $\mathbf{L}(k, \hat{\mathbf{x}})$ satisfies the stable criterion, by Lemma 1. The stable criterion in turn implies that each column indexed by a monomial $\mathbf{x}^\mathbf{j}$ for $|\mathbf{j}| = k$ is a pivot column, hence there is an entry in G_k with leading term $\mathbf{x}^\mathbf{j}$. Now from the argument of Lemma 3 each element of $\mathcal{I}$ is a sum $g + h$ where g is in the span of the elements of F_k and h is a finite linear combination of $\mathbf{x}^\mathbf{j}$ where $|\mathbf{j}| > k$. If $h \neq 0$ then as noted above the leading term of $g + h$ is divisible by the leading term of an entry of G_k, while if $h = 0$ then $g + h = g$ is a linear combination of entries of G_k so the leading term is the leading term of some entry of G_k because of the echelon structure of R. ∎

It needs to be emphasized that Theorem 1 holds only because we are working with a zero dimensional local ring. In general one cannot expect to be able to compute Gröbner bases using RRREF, even for zero dimensional rings.

It is a very classical result, eg. [1], that any finite dimensional $\mathbb{C}$-algebra of $\mathbb{C}$-dimension m is isomorphic to an $m \times m$ matrix algebra. This isomorphism is given by assigning an element f to the multiplication matrix M_f. These matrices have been used extensively by a number of authors in system solving, for example [13, 17, 3, 8] and the many references in those works. In this case we notice the form the presentation above is essentially a local border basis, so we can directly compute the multiplication, or, as some authors more accurately call them, transpose multiplication matrices M_f.

Example 1 continued: From the presentation above we have a $\mathbb{C}$-basis of the local ring consisting of $1, x, x^2$. One can then directly read off

$$M_x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad M_y = \begin{bmatrix} 0 & \frac{7}{8} & 0 \\ 0 & 0 & \frac{7}{8} \\ 0 & 0 & 0 \end{bmatrix}$$

The relations given above are easily checked, eg. $M_y = \frac{7}{8}M_x, M_x^3 = 0$ ∎

Note that the multiplication matrices above have only the eigenvalue $\lambda = 0$, this is because in a finite dimensional local ring every element M_f in the maximal ideal is nilpotent. We also point out that the construction of the matrices $\mathbf{L}(k, \hat{\mathbf{x}})$ shifts the origin to $\mathbf{0}$, whether or not $\hat{\mathbf{x}} = 0$. We then get

Proposition 1 *Let $A = \mathbb{C}[x_1, \ldots, x_t]/\langle f_1, \ldots, f_t\rangle$ and $\mathcal{O}_{\hat{\mathbf{x}}} = \mathcal{O}(A)_{\hat{\mathbf{x}}}$ be the local ring at $\hat{\mathbf{x}} = \mathbf{0}$. Assume that $\mathcal{O}_{\hat{\mathbf{x}}}$ is finite dimensional as a $\mathbb{C}$-algebra with $\mathbb{C}$-basis $g_1, \ldots g_m$. The following identities must be satisfied:*

$$M_{x_i} M_{x_j} - M_{x_j} M_{x_i} = \mathbf{0}, \qquad \text{all } i, j \tag{2}$$

$$M_{x_1}^{j_1} M_{x_2}^{j_2} \cdots M_{x_s}^{j_s} = \sum_{i=1}^{m} c_i M_{g_i} \qquad j_\alpha \geq 0, \ \text{some } c_i \in \mathbb{C} \tag{3}$$

$$f_\alpha(M_{x_1}, \ldots, M_{x_s}) = 0, \qquad \alpha = 1, \ldots, t, \ \text{when } \hat{\mathbf{x}} = \mathbf{0} \tag{4}$$

$$\lambda_{i,1} = \cdots = \lambda_{i,m} = 0, \qquad \lambda_{i,j} \text{ eigenvalues of } M_{x_i} \tag{5}$$

Note if $\hat{\mathbf{x}} \neq \mathbf{0}$ then (3) would need to be replaced by $f_\alpha(M_{x_1} + \hat{x}_1 I_m, \ldots, M_{x_s} + \hat{x}_s I_m) = 0$ where $\hat{\mathbf{x}} = (\hat{x}_1, \ldots, \hat{x}_s)$.

Example 2: Consider the system

$$\begin{aligned} f_1 &= 2x^2 - x - x^3 + z^3 \\ f_2 &= x - y - x^2 + xy + z^2 \\ f_3 &= xy^2 z - x^2 z - y^2 z + x^3 z \end{aligned}$$

Here $\hat{\mathbf{x}} = (0, 0, 0)$ is an isolated zero of multiplicity 5, and there are also two other simple isolated zeros but the ideal is not zero dimensional as the entire line $z = 0, x = 1$ is a solution of the system $\mathcal{I} = \langle f_1, f_2, f_3\rangle$.

A standard implementation of RREF is sufficient for reducing $\mathbf{L}(5, \hat{\mathbf{x}})$ to reverse reduced row echelon form, a 51×56 matrix. The associated $\mathbb{C}$-basis is $1, x, y, z, xz$ and the nil-index is 4, that is all rows after the 30^{th} have zero entries except for a 1 as the $i, i+5$ entry, that is, all monomials of degree 5 or greater represent zero in the local ring.

The only non-trivial equations come from rows 3,4,5,14,15 and 30. These are

$$-xz + y^2, \ -x - xz + yz, \ x - y + z^2, \ -xz + yz^2, \ -x + z^3, \ -xz + z^4. \tag{6}$$

Further, every monomial which is not in the $\mathbb{C}$-basis or a leading term of one of the monomials above is zero, in particular, x^2, xy, y^3 and z^5. These last polynomials together with those in (6) form a global Gröbner basis for the local ring $\mathcal{O}_{\hat{\mathbf{x}}}$.

On the other hand it is easy to compute from our RREF matrix that

$$M_z = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By direct calculation one finds $M_z^4 \neq 0$ but $M_z^5 = 0$. This, together with equations $M_x = M_z^3, M_y = M_z^2 + M_z^3$ obtained from the equations (6) give a quick representation of the local ring.

Thus, even though the original system is not zero dimensional, this method easily computes the structure of the the local ring of a zero dimensional local ring.

4 The Exact Case with Inexact Arithmetic

When reducing the local array as in the last section, in general the reduced row echelon form algorithm using floating point numbers will not give a correct result in even if one starts with an exact system. In Example 1 starting from $\mathbf{L}(3, \mathbf{0})$ seems to give adequate results using MAPLE, however one should, in theory, get the same results from $\mathbf{L}(7, \mathbf{0})$, but that does not happen using default precision arithmetic.

To handle this issue we use an Approximate Reduced Row Echelon form. A similar algorithm has been used in [15] In [8] an approximate LU decomposition is achieved by Gaussian elimination with complete pivoting, but complete pivoting is not appropriate here.

More precisely, because of the need to start at the lower right, the following *approximate reverse reduced row echelon form* algorithm is used which has worked reasonably well in our examples.

Algorithm ARRREF

- **Input:** $m \times n$ matrix M, tolerance ε.
- Apply the SVD decomposition to get $M = Q_1 \Sigma Q_2^\top$. Calculate the approximate rank r by counting the number of diagonal entries of Σ which are greater than ε. The approximate row space of M is then spanned by the first r rows of $Q_2^\top$, call the $r \times n$ matrix of these rows A.

- We can assume the right hand, n^{th} column of M, hence A, is not $\mathbf{0}$. Initialize a matrix B to be the n^{th} column. Starting with the $j = n - 1^{st}$ column and proceeding leftward we can check, using SVD or rank revealing, that the j^{th} column is approximately independent of the space spanned by columns $j+1, \ldots, n$, i.e. the approximate rank increases by 1 when this column is added. If the j^{th} column is approximately independent then add this column as the first (leftmost) column of B. When done B will consist of independent columns of A. B should be reasonably well conditioned.

- **Output:** $R = B^{-1}A$. (B^{-1} is just the usual inverse.) and a list of indicies of the pivot columns.

Note that if some column is a *pivot column*, i.e. one of the columns placed in B, then all dependence relations are with columns to the left. Thus, in R, all entries above and to the right of the pivots should be numerically zero. This algorithm could fail, so the results should be checked, a small adjustment of the tolerance may help.

Example 3: Here is a less trivial example, the 3×3 system

$$\begin{aligned} f_1 &= x^3 + 5z \\ f_2 &= x^2y + y^4 \\ f_3 &= z + 7xy^4 - 6y^5 \end{aligned}$$

This is similiar to [5, Example 1] with an extra, non essential, variable and terms added to make the presentation of $\mathcal{O}_{\hat{\mathbf{x}}}, \hat{\mathbf{x}} = \mathbf{0}$ less obvious. The Hilbert function is $\{1, 2, 3, 2, 2, 1, 1, 0, \ldots\}$ and so $\mathbf{L}(7, \mathbf{0})$ is the first local array to satisfy the stable criterion using a tolerance of 10^{-6} in ARRREF. $\mathbf{L}(7, \mathbf{0})$ is a 252×120 matrix while the ARRREF matrix $\mathbf{R}$ is 108×120 giving multiplicity 12. All the numerically non-zero entries in $\mathbf{R}$ are very close to $1, \pm 5$, $\pm\frac{1}{6}$ or $\pm\frac{35}{6}$. Since, given the coefficients of the system these values make sense, we assume that we have these exact values. It should be noted that attempting to do this calculation using a numerical version of RREF, may not work, MAPLE RREF, for example gives a matrix of rank 111, that is, multiplicity only 9.

For comparision, a presentation of $\mathcal{O}_{\mathbf{0}}$ was also calculated by SINGULAR using the local ordering `Ds`. Our assumption above about exact values is confirmed. The presentations given by these two methods differ. For instance, there are 9 monomials in the $\mathbb{C}$-basis given by both methods, namely $1, x, y, x^2, xy, y^2, xy^2, y^3, xy^3$. The other three monomials differ. The monomials are given as the left hand terms in the chart below, the right hand of these equations gives these basis monomials in terms of those of the other basis.

NLR method		SINGULAR	
z	$= 6y^5 - 210y^6$	y^4	$= -x^2y$
x^2y	$= -y^4$	y^5	$= \frac{1}{6}z + \frac{35}{6}yz$
yz	$= 6y^6$	y^6	$= \frac{1}{6}yz$

In fact the $\mathbb{C}$-basis and presentation given by the ALR method is exactly that given by SINGULAR for the global ordering `Dp` applied to the local ring $\mathcal{O}_{\mathbf{0}}$. This presentation is $\mathbb{C}[x, y, z]/J$ where J is the ideal generated by

$$\{x^3 + 5z,\ y^4 + x^2y,\ z^2,\ xz,\ y^2z,\ 6x^2y^2 + 35yz + z\}$$

All these relations are already given as rows of the ARRREF result $\mathbf{R}$, although many additional redudundant relations, 102 of them to be exact, are also given. Note, in particular that $y^7 = 0$ and in fact all monomials of degree 7 and above are zero. An advantage of the SINGULAR local $\mathbb{C}$-basis is that one can read off the Hilbert function from the basis, while this is not obvious from the global basis.

To represent $\mathcal{O}_{\mathbf{0}}$ as a matrix algebra it is enough to give M_x, M_y as $M_z = -\frac{1}{5}M_x$. These matrices are easily calculated from the presentation above and satisfy the conditions of Proposition 1.

5 An inexact approximation of a multiple zero

The method in [5] and here was designed to be insensitive to small pertubations. As observed in the introduction, a small change in the calculated zero did not change the multiplicity stucture, in this section we give an example to show that this is true as well for a small pertubation of the coefficients.

Example 1 continued: Random errors of magnitude 10^{-4} are added to all the coefficients of total degree less than or equal to four, including coefficients that were originally zero, of the system in Example 1. Further, a random error of the same magnitude is applied to the zero $\hat{\mathbf{x}} = \mathbf{0}$. The resulting system was given to the homotopy continuation package BERTINI [2]. There were no multiple or singular zeros found, even when the default tolerances are loosened. In particular, the condition numbers of the Jacobian matrix are all less than 10^4, well under the given or default tolerance for these numbers.

However, loosening the tolerance in the ARRREF algorithm to 10^{-3} one gets a result which, when rounded to 5 decimal places, differs in only three entries from the exact $\mathbf{R}$

of Example 1, and these differences are small:

$$\mathbf{R} = \begin{bmatrix} 0 & -0.87503 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.87498 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.76559 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus one recovers the same $\mathbb{C}$-basis $\{1, x, x^2\}$ for the local ring at the origin , and essentially the same relations. Moreover calculating the multiplication matrices from this $\mathbf{R}$ gives the same matrix for M_x to 5 decimal places and

$$M_y = \begin{bmatrix} 0 & 0.87503 & 0 \\ 0 & 0 & 0.87498 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus one could deduce the same structure as in the exact Example 1, i.e. $y = -0.87503x$, $x^3 = 0$

It should be noted, however, that the equations of Proposition 1 are only approximately true. For example $\|My - 0.87503Mx\| \approx 9.3 \times 10^{-6}$ and $\|Mx^3\| \approx 7.4 \times 10^{-4}$. Moreover, the eigenvalues of Mx, My are, instead of being zero are of the order of 10^{-2}. It is for this reason that we say the matrices M_x, M_y generate a numerical local ring.

More generally when given an inexact system or a system of analytic functions the presentation $\mathcal{O}_{\hat{\mathbf{x}}} = \mathbb{C}[x_1, \ldots, x_s]/\mathcal{I}$ may not make rigorous sense. Instead the set of matrices $\{M_{x_1}, \ldots, M_{x_s}\}$ which approximately generate a local $\mathbb{C}$-algebra will be called the *numerical local ring*.

6 Expanding the local region

In the previous section we had a perturbed system which, according to Bertini, had distinct zeros close to $\hat{\mathbf{x}} = \mathbf{0}$. By using a loose tolerance in ARRREF we were able to recover the structure of a local ring of a nearby system with a multiple zero. In this section we go the opposite direction. Given a system with a numerical multiple zero we construct a nearby system with distinct zeros. This can be useful, for example, when a numerical solution of an exact system is refined by LVZ deflation [10] distinct zeros may

be included in a numerically multiple zero. This will be illustrated by examples in the next section.

By [5, Lemma 1] with exact arithmetic the co-rank $\mathbf{L}(k, \hat{\mathbf{x}})$ stabilizes at the multiplicity for all k larger than the nil-index. Numerically, however, this may not happen, for using a larger k and looser tolerance in ARRREF one may get a larger co-rank for a stable $\mathbf{L}(k, \hat{\mathbf{x}})$, i.e. the system may have a larger numerical multiplicity at $\hat{\mathbf{x}}$. Thus the numerical multiplicity may depend on both the choice of tolerance and the choice of k.

One can then construct multiplication matrices from the ARRREF data as above. But, working numerically, Proposition 1 will not hold exactly, in particular the eigenvalues of the M_{x_i} may not be zero. Since we are describing the local ring in global terms, we may use the methods of [13, 17] to find the zeros of this ring. The one technicallity is that since the multiplication matrices only approximately commute, we have to use a numerical method to approximate the common eigenvectors . The following iterative least squares method appears to be satisfactory for this purpose.

Common Eigenvector Calculation (CEC): Let $M_{x_1}, \ldots M_{x_s}$ be the multiplication matrices. For random floating point numbers $r_1, \ldots, r_s$ let $M = r_1 M_{x_1} + \cdots + r_s M_{x_s}$. Use the eigenvectors of M, rather than any M_{x_i}. (This is apparently a well known trick, for example it is used in [4, 8]).

For each normalized eigenvector $\mathbf{v} = [1, v_2, \ldots, v_m]^\top$ of M solve the quadradic least squares problem

$$\begin{bmatrix} M_{x_1}\mathbf{u} - \mu_1\mathbf{u} \\ M_{x_2}\mathbf{u} - \mu_2\mathbf{u} \\ \vdots \\ M_{x_s}\mathbf{u} - \mu_s\mathbf{u} \\ u_1 - 1 \end{bmatrix} = \mathbf{0}$$

in variables $u_1, \ldots, u_m, \mu_1, \ldots, \mu_s$ where $\mathbf{u} = [u_1, u_2, \ldots, u_m]^\top$. This seems best solved by one or more iterations of Gauss-Newton using inital values $\mathbf{u} = \mathbf{v}$, μ_i the first coordinate of $M_{x_i}\mathbf{v}$, $i = 1..s$.

The desired zero is given by $\hat{\mathbf{x}} = [\mu_1, \ldots \mu_s]^\top$. ∎

Thus we have a method for finding distinct zeros close to a given approximate zero: first calculate $\mathbf{L}(k, \hat{\mathbf{x}})$ for $k = 1, 2, \ldots$ until two consecutive values of the numerical co-rank, using a loose tolerance, agree at a high enough number. Then, in analogy with Lemma 1 our stable criterion will hold numerically for the smaller of the consecutive values. Apply the ARRREF with this tolerance and extract the multiplication matrices, then apply

the CEC above to calculate zeros near $\hat{\mathbf{x}}$. Finally these zeros should be further analyzed to decide if they should be treated as distinct or part of a multiple zero. Since these multiplication matrices are generators of a *numerical local ring*, we will call this the NLR method.

7 Two dimensional polynomial Examples

In this section we look at some examples.

Example 1 continued: Setting a loose tolerance, we get a numerical Hilbert function that starts $1, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0$. Thus $\mathbf{L}(8, \mathbf{0})$ has rank deficiency 6 and the stable criterion is satisfied so we expect to find 6 zeros near $\mathbf{0}$. We calculate the multiplication matrices with respect to $\mathbb{C}$-basis $\{1, x, y, x^2, y^2, x^3\}$ and find the norm of the commutators $\|M_x M_y - M_y M_x\| \approx 0.29$ which is quite large. Applying our CEC algorithm above, however, gives three points very close to zero, the largest norm being about 10^{-7}. The other 3 points turn out to be real and are given below and compared with actual solutions of the exact system by Bertini and Singular.

Point	Calculated Point	Nearby actual Solution
(a)	$\begin{bmatrix} 0.1034529096 + 0.0\,i \\ 0.0911846931 + 0.0\,i \end{bmatrix}$	$\begin{bmatrix} 0.10345197 \\ 0.091183856 \end{bmatrix}$
(b)	$\begin{bmatrix} 0.4702795691 + 0.0\,i \\ 0.6442715385 + 0.0\,i \end{bmatrix}$	$\begin{bmatrix} 0.45801998 \\ 0.69681723 \end{bmatrix}$
(c)	$\begin{bmatrix} -0.4468529986 + 0.0\,i \\ -0.8391717555 + 0.0\,i \end{bmatrix}$	no nearby solution

The first point, which is close to $\mathbf{0}$ is given very accurately, the second is somewhat less accurate but Newton's method converges quickly to the nearby solution. The third, however does not correspond directly to an actual solution.

This illustrates typical behavior of the NLR method. It gives high accuracy for points very close to the center $\hat{\mathbf{x}}$, it is less accurate for points that are close but further out, and generally will not give good results for points that are a significant distance from

the center. In fact point (c) above gives an example showing that points calculated by this method may not be approximations of solutions of the original system.

In this example point (a) is approximately distance 0.13 from $\mathbf{0}$, point (b) is distance 0.7 from the center, which is near the limit of this method. It can be noted that there are a pair of complex solutions a distance 1.08 from the center with real part close to this point. There are 5, counting multiplicity, points closer to $\mathbf{0}$ and M_x, M_y are 6×6 matrices. Since a real matrix cannot have an odd number of complex eigenvalues it would not be possible to acheive one of these complex solutions without the other. But the rank deficiency does not increase until $\mathbf{L}(12, \mathbf{0})$ and by then the commutator $MxMy - MyMx$ is large and decent results are impossible. ∎

Example 4: Consider the following complex polynomial

$$f(z) = \left(\frac{11}{5}\,z - 1/10 + 1/10\,i\right)^2 \left(11\,z - \frac{49}{100}\right)\left(\frac{1}{3}\,z - \frac{5}{3} - \frac{68}{33}\,i\right)\cdot \\ \left(\frac{1}{3}\,z - \frac{26}{33} - \frac{13}{33}\,i\right)\left(\frac{1}{3}\,z + \frac{65}{33} + \frac{5}{33}\,i\right)\left(\frac{1}{37}\,z - \frac{6}{37} + \frac{36}{407}\,i\right)$$

Then let $h_1 = \operatorname{Re} f(x+iy)$ and $h_2 = \operatorname{Im} f(x+iy)$. The real system $H = \left[\begin{smallmatrix} h_1 \\ h_2 \end{smallmatrix}\right]$ in variables x, y has 49 total zeros, counting multiplicity. Next we replace the rational coefficients by 10 digit floating point approximations and solve the system $\tilde{H}$ by the homotopy continuation package PHCpack [18]. PHCpack again finds 49 solutions of which 47 are considered regular and 2 are clustered using default settings in blackbox mode.

Example 4a Here we use the NLR method to find and classify the nine zeros of the numerical system $\tilde{H}$ near the origin. With $\hat{\mathbf{x}} = (0, 0) = \mathbf{0}$ and a medium tolerance the stable criterion on $\mathbf{L}(10, \hat{\mathbf{x}})$, a 110×66 matrix with approxi-rank 57 is satisfied. A calculation of the ARRREF form leads to the multipliciation matrices with commutator norm $5 * 10^{-9}$. Using the CEC we get 9 zeros from the 9 eigenvectors of a random linear combination of M_x, M_y which compare within two significant digits to PHCpack regular solutions 1,34,11,31,28,12,18,30 and 47 respectively. The main difference is that the 9 near zeros are scattered through the 49 zeros given by PHCpack where the NLR method only gives the near zeros.

Refinement by Newton's method, no deflation, suggests that zeros 1 and 3 are the same double zero as are zeros 2 and 4. Zero 5 is a simple zero, but zeros 6 through 9 all appear to refine to the same solution of multiplicity 4. Now deflation can be used on the multiple zeros and we conclude the zeros of this sytem near the origin, with multiplicities

are

$$\hat{\mathbf{x}}_1 = \begin{bmatrix} 0.045000000 + 0.0227272727\,i \\ -0.027272727 + 0.000454545\,i \end{bmatrix} \qquad m = 2$$

$$\hat{\mathbf{x}}_2 = \begin{bmatrix} 0.045000000 - 0.0227272727\,i \\ -0.027272727 - 0.000454545\,i \end{bmatrix} \qquad m = 2$$

$$\hat{\mathbf{x}}_3 = \begin{bmatrix} .0445454545 \\ 0 \end{bmatrix} \qquad m = 1$$

$$\hat{\mathbf{x}}_4 = \begin{bmatrix} .0454545455 \\ -.0454545455 \end{bmatrix} \qquad m = 4$$

These points and multiplicities agree with the solution of the exact system H. ∎

Example 4b: Of the 49 zeros of the example $\tilde{H}$ of this subsection PHCpack identifies two pairs of clustered zeros. In fact, there are more, such as the ones above. In this example we analyze one of the identified clusters, PHCpack solutions 5 and 6 which agree to 10 significant digits.

Starting with solution 5,

$$\hat{\mathbf{x}}_5 = \begin{bmatrix} 1.20462113364444 + 0.613535853038710\,i \\ 0.568282329506649 - 1.15901522965055\,i \end{bmatrix}$$

Deflate using PHCpack with tolerance on numerical rank set at $1.0\mathrm{E}-03$. The result is

$$\tilde{\mathbf{x}}_5 = \begin{bmatrix} 1.20461321453912 + 0.602692583945157\,i \\ 0.579125598432050 - 1.15902314734559\,i \end{bmatrix}$$

This is suprisingly far from both solutions 5 and 6.

The reason is $\mathbf{L}(k, \mathbf{0})$ first becomes stable at $k = 4$ with $m = 3$. So there is another near zero affecting the deflation. To get increased accuracy we use $\mathbf{L}(5, \tilde{\mathbf{x}}_5)$ and directly, without further refinement, produce the following three zeros

$$\hat{\mathbf{y}}_1 = \begin{bmatrix} 1.20444079658657555 + .613795791160425930\,i \\ .568022391235419422 - 1.15919556593683582\,i \end{bmatrix}$$

$$\hat{\mathbf{y}}_2 = \begin{bmatrix} 1.20465162929802649 + .613476260252905803\,i \\ .568197556363474355 - 1.15898311383042696\,i \end{bmatrix}$$

$$\hat{\mathbf{y}}_3 = \begin{bmatrix} 1.20408938594172610 + .590909745035199552\,i \\ .590908437362156924 - 1.15954697659234785\,i \end{bmatrix}$$

Note $\hat{\mathbf{y}}_1, \hat{\mathbf{y}}_2$ are close to the clustered solutions 5,6 but $\hat{\mathbf{y}}_3$ is near PHCpack solution 7.

With tolerance set at 0.005 we find the approxi-rank of the Jacobian matrix of ordinary Newton's method (no deflation) is 2 at $\hat{\mathbf{y}}_1$ which suggests that it is a simple zero, while the Jacobian matrix has approxi-rank 0 at the other two points, suggesting that they are approximations of a double zero. Using PHCpack to do deflation, still with tolerance on numerical rank at $1.0\mathrm{E}-05$, from $\hat{\mathbf{y}}_1$ gives

$$\tilde{\mathbf{y}}_1 = \begin{bmatrix} 1.20454545800870 + 0.613636354355985\, i \\ 0.568181828189372 - 1.15909090528630\, i \end{bmatrix}$$

Finally note, see below, that the deflated answer $\tilde{\mathbf{y}}_1$ agrees with exact solution of the rational system to 9 decimal places. It important to use an initial point of deflation which avoids unwanted points of the cluster. ∎

8 An analytic Example

Although the motivation for the NLR method comes from the study of polynomial systems, this method may have a significant application in analytic systems where it is often not possible to have a solution method which gives all solutions. This system was given by Griewank and Osborne [6] in 1983.

Consdier the system $G = [g_1, g_2, g_3]^\top$ is given by

$$\begin{aligned} g_1 &= 0.5x^2\cos(10x) + x\sin(10y) + 10x\sinh(z) \\ g_2 &= \cos(5x) + \sin(5y) - 1 \\ g_3 &= \cosh(5x) + 5\sinh(z) - 1 \end{aligned}$$

Example 5a: Griewank and Osborne were specifically interested in minimization problems and, hence, real solutions. They identified the origin $(0, 0, 0)$ as a multiple isolated solution with a nearby simple zero at approximately $(.12, .033, -.035)$. In fact the NLR method found an additional 4 imaginary solutions close by that the author had not previously known.

With loose tolerance the stable criterion is satisfied by $\mathbf{L}(7, \hat{\mathbf{x}})$ with co-rank $m = 7$ for $\hat{\mathbf{x}} = \mathbf{0} = (0, 0, 0)$. Applying ARRREF to $\mathbf{L}(7, \mathbf{0})$, a 252×120 matrix, returns a 113×120 matrix R which consists of a 113×7 matrix appended with the identity I_{113}.

It is then easy to extract the 7×7 multiplication matrices M_x, M_y and M_z for $\mathbb{C}$-basis $1, x, y, z, x^2, xy, xz$.

One calculates the commutators $\|M_i M_j - M_j M_i\| < .09$ for $1 \leq i < j \leq 3$.

The CEC algorithm returns $\mathbf{0}$ for the first two eigenvectors of a random linear combination of M_x, M_y, M_z, which is no suprise as the origin is known to be a multiple zero. The other 5 eigenvectors give (rounded)

$$\begin{bmatrix} .06446 \pm .2580i \\ -.1526 \pm .1379i \\ .1416 \mp .06214i \end{bmatrix}, \quad \begin{bmatrix} .1159 \\ .03280 \\ -.03453 \end{bmatrix}, \quad \begin{bmatrix} .1296 \mp .08079i \\ .02658 \pm .04998i \\ -.02391 \mp .05458i \end{bmatrix}$$

with residues $.7 * 10^{-3}, .2 * 10^{-6}, .5 * 10^{-5}$ respectively.

Newton refinement gives the following values (rounded) with residues $< 10^{-9}$:

$$\begin{bmatrix} .06534 \pm .26056i \\ -.15336 \pm .14012i \\ .14398 \mp .06352i \end{bmatrix}, \quad \begin{bmatrix} .11591 \\ .03281 \\ -.03453 \end{bmatrix}, \quad \begin{bmatrix} -.12972 \pm .08182i \\ .02732 \mp .05071i \\ -.02350 \pm .05531i \end{bmatrix}$$

with smallest singular value of the Newton Method Jacobian being $1.1, 0.10, 0.16$ respectively. Thus we can conclude that these 5 zeros are simple zeros and the zero at the origin is of multiplicity 2. ∎

Example 5b By inspection it is easily seen that $\hat{\mathbf{x}} = [0, \frac{\pi}{5}, 0]^\top$ is also a solution to the system G. A calculation using $\mathbf{L}(k, \hat{\mathbf{x}})$ for various k, ε shows that the stable criterion holds for $k = 5$ giving co-rank $m = 3$ for even tight tolerance. One is tempted to call $\hat{\mathbf{x}}$ a zero of multiplicity 3. However, applying ARRREF to $\mathbf{L}(5, \hat{\mathbf{x}})$, where $\hat{\mathbf{x}} = [0, 0.6283185, 0]^\top$, extracting the multiplication matrices and applying CEC to the first eigenvector we get a zero of the NLR $\hat{\mathbf{y}}$ which suggests zero $\hat{\mathbf{x}} + \hat{\mathbf{y}}$ of the original system G:

$$\hat{\mathbf{y}} = \begin{bmatrix} .0099595 \\ -.00024748 \\ -.00024758 \end{bmatrix}, \quad \hat{\mathbf{x}} + \hat{\mathbf{y}} = \begin{bmatrix} .0099595 \\ .62807 \\ -.0024758 \end{bmatrix}.$$

The other two zeros give an offset of $\hat{\mathbf{x}}$ of essentially $\mathbf{0}$ so one should conclude that $\hat{\mathbf{x}}$ is a zero of multiplicity 2 while $\hat{\mathbf{x}} + \hat{\mathbf{y}}$ is a zero of multiplicity 1. Further Newton refinement using multiple precision on the exact original system supports this conclusion.

References

[1] G. Birkhoff, S. Mac Lane, *A survey of Modern Algebra*, Macmillan, 1941.

[2] D. Bates, J. Hauenstein, A. Sommesse and C. Wampler II, *Bertini: Software for Numerical Algebraic Geometry*, Software available at http://www.nd.edu/sommese/bertini.

[3] D. Bondyfalat, B. Mourrain, V. Pan, *Solution of a polynomial system of equations via the eigenvector computaion*, Lin. Alg. and its Appl. 319, pp. 193-209, 2000.

[4] D. Cox, J. Little, D. O'Shea, *Using Algebraic Geometry.* Springer Verlag, 1998.

[5] B. H. Dayton and Z. Zeng,*Computing the Multiplicity Structure in Solving Polynomial systems.*, Proc. of ISSAC '05, ACM Press, pp. 116–123.

[6] A. Griewank and M.R. Osborne, *Analysis of Newton's Method at irregular singularities.* SIAM J. Numer. Anal. 20(4), pp. 747-773, 1983.

[7] G.-M. Greuel and G. Pfister, *A Singular Introduction to Commutative Algebra.* Springer Verlag, 2002.

[8] I. Janovitz-Freireich, L. Ronyai, A. Szanto, *Approximate Radical of Ideals with Clusters of Roots.* Proc of ISSAC '07, ACM press, pp. 146–153.

[9] A. Leykin, J. Verschelde, and A. Zhao, *Evaluation of Jacobian matrices for Newton's method with deflation to approximate isolated singular solutions of polynomial systems.* In D. Wang and L. Zhi, editors, SNC2005 Proceedings, International Workshop on Symbolic-Numeric Computation, pp. 19–28, 2005.

[10] A. Leykin, J. Verschelde, and A. Zhao, *Newton's method with deflation for isolated singularities of polynomial systems.* Theoretical Computer Science 359 (2006), no. 1–3, pp. 111-122.

[11] T. Y. Li and Z. Zeng, *A rank-revealing method with updating, downdating and applications.* to appear: SIAM J. Matrix Anal. Appl.

[12] F. S. Macaulay, *The Algebraic Theory of Modular Systems.* Cambridge Univ. Press, 1916.

[13] H. M. Möller and H. J. Stetter, *Multivariate polynomial equations with multiple zeros solved by matrix eigenproblems*. Numer. Math., 70 (1995), pp. 311–329.

[14] T. Mora, *Solving Polynomial Equation Systems II*. Encyclopedia of Mathematics and its Application, 99, Cambridge University Press, Cambridge, 2005.

[15] R. Scott, *Approximate Gröbner bases – a backwards approach*. Master of Science Thesis, Greg Reid supervisor, Graduate Program in Applied Mathematics, University of Western Ontario, London, Ontario, Canada, 2006.

[16] A. J. Sommese and C. W. Wampler II *The Numerical solution of Systems of Polynomials Arising in Engineering and Science*. World Scientific Publishing Co., 2005.

[17] H. J. Stetter, *Numerical Polynomial Algebra*. SIAM, 2004.

[18] J. Verschelde, *Algorithm 795: PHCpack: A general-purpose solver for polynomial systems by homotopy continuation*. ACM Trans. Math. Softw., 25(2) pp 251–276, 1999. Software available at http://www.math.uic.edu/∼jan. (July 2005 version used in this paper.)