Numerical Local Rings of Analytic Systems via Differentials

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Given a system of complex valued functions $f_1, \ldots, f_n$ holomorphic in a neighborhood of a point $\hat{\mathbf{x}} \in \mathbb{C}^s$ the *numerical local ring method* [2] attempts to understand the structure of isolated zeros near $\hat{\mathbf{x}}$. This paper gives a new approach to this method based on differential functionals. The advantages of this approach are that, first, the behavior of this method is more transparent and, secondly, the point $\hat{\mathbf{x}}$ need not be a zero of the system for the method to be effective.

1 Introduction

This report originated as an attempt to write a journal article giving a theoretical foundation for the Numerical Local Ring (NLR) method in [2]. I had found a new approach based on differential functionals which appeared to make the method more transparent. Unfortunately this transparency has illuminated several unresolved issues. So I am only giving a progress report at this time. Since the new algorithm is an impovement, I am making this report available to those with an interest in this method.

The *numerical local ring* method was introduced in [2] in an attempt to analyze a numerically multiple point that might in fact be a cluster of nearby points of smaller multiplicities. In that paper the method was justified by a loose analogy to the local ring of a point and numerical experiments. The method in [2] used the somewhat controversial ARRREF (approximate reverse reduced row echelon form), produced a system of possibly inconsistent equations and then applied the Möller-Stetter method to a set of multiplication matrices that did not strictly commute. Using a least square refinment I was often able to produce approximations of near zeros of the system which often helped understand the results of [3] in the presence of a zero cluster near $\hat{\mathbf{x}}$.

The present point of view is that there is an actual ring that I am attempting to calculate, namely $\mathcal{H}(\mathcal{U} \cap V_F)$. Given an analytic system $F = [f_1, \ldots, f_n]^\top$ on $\mathbb{C}^s$ and an Euclidean open

set $\mathcal{U} \subseteq \mathbb{C}^s$, $\mathcal{H}(\mathcal{U} \cap V_F)$ is the algebra of holomorphic functions on $\mathcal{U} \cap V_F$ where V_F is the variety defined by F, see [10, Definition 3.8.1]. Assuming only isolated zeros in the open set $\mathcal{U}$ this algebra is the product of finitely many local rings. I show how to construct $\mathcal{H}(\mathcal{U} \cap V_F)$ given accurate information about the zeros and local rings. What one would like is a numerical method that would start with the system F and $\mathcal{U}$ and produce $\mathcal{H}(\mathcal{U} \cap V_F)$.

The philosophy of this new approach is as follows. The Macaulay local array of the system F consists essentially of truncations of monomial multiples of the original equations given as Taylor expansions about the zero $\hat{\mathbf{x}}$. Truncation does not greatly effect the value of these functions near $\hat{\mathbf{x}}$ but does alter the values further away. The cumlative effect is to kill all zeros of the system but $\hat{\mathbf{x}}$ and thus create a system with only the one zero, but which agrees with the original system to high precision near that zero, in particular giving the same local structure. From the differential functional point of view each zero corresponds to a subspace of differential functionals, the dimension of which is the multiplicity of the zero. These functionals vanish on all members of the ideal of the system. As noted in [3], and earlier by Macaulay [7], the functionals associated with the point $\hat{\mathbf{x}}$ appear as vectors in the nullspace of the local array. The process of truncation causes those differentials not associated with the point $\hat{\mathbf{x}}$ to not vanish on the truncated rows, but in the case of zeros nearby $\hat{\mathbf{x}}$, since the change is slight, these differentials when applied to the truncated members of the ideal still nearly vanish. The philosophy of the current method is to attempt to recover these differential functionals from the approxi-nullspaces of the local arrays. As this report shows, I am only partially successful.

This new method replaces the slow ARRREF method with a much faster reduction method on a small space of differentials. Like the earlier method, this one relies on the local arrays of a point. In [2] the analogy with the exact method, whereby it is enough to calculate the local array of any degree higher than the depth, as defined in [3], was used. Here I note that greater accuracy can be achieved by using local arrays of as high a degree as practical. Using higher degrees it is no longer necessary that $\hat{\mathbf{x}}$ be itself a zero of the system. This greater accuracy has a cost of more computation. Therefore, although theoretically more pleasing, the new method may not be computationally superior to the original method described in [2].

In [3, 2] the exposition was limited to polynomial systems but included an example to show that the method gave information on analytic systems also. In this paper I focus on analytic systems, of which polynomial systems form a subset, for two reasons. There are several technical issues with holomorphic systems that are not present with polynomial systems, therefore I need to consider these issues in our development of these ideas. This is especially true of the main technical lemma, Claim 1. There is currently good software to find all zeros of moderate size polynomial systems. Although the output of PHCpack or Bertini [1, 11] does not explicitly identify clusters, it is certainly possible to search the output for clusters. But analytic systems often have infinitely many solutions and finding zeros nearby a multiple zero can often be difficult. Therefore this method seems to have more potential applications for analytic systems. I note, however, that the various types of local rings I will discuss, even for analytic systems, are all polynomial rings.

In §2 I review the local array from [3] where it was called the multiplicity matrix. I note that from [2] a basis for the ideal of a local ring at a point can easily be extracted from this matrix.

In §3 I discuss differential functionals and explain how to extract multiplication matrices or a

basis for the ideal of a system of equations from certain matrices associated to the differential functionals. I give an example showing how to construct $\mathcal{H}(\mathcal{U} \cap V_F)$ given all zeros in a neighborhood using the method in [3] together with the results of this section. B. Mourrain [8] has previously used differential functionals to analyze local behavior of polynomial systems.

In §4 I give the technical Claim 1 showing that as the degree of the local array increases differential functionals associated with zeros near the center $\hat{\mathbf{x}}$ are revealed. In particular this shows that it is useful numerically to use local arrays of as large a degree as is computationally feasible, even though the dimension of the right null space may stabilize. I show that the presence of differential functionals is often indicated by small singular values of the local matrix. I give an application, Example 3, showing how one can approximate $\mathcal{H}(\mathcal{U} \cap V_F)$, for some unspecified $\mathcal{U}$, without knowing the zeros and their multiplicity in advance. I call the result of this approximation the *numerical local ring*.

In §5 I formalize as an algorithm the method of Example 2, and recall from [2] one of the original applications of this method to an analytic system of Griewank and Osborne [4].

2 The Local Array

For the reader's convenience we will summarize the construction in [3, 2] here. Let $f_1, f_2, \ldots, f_n$ be a list of n functions in s variables $x_1, \ldots, x_s$ holomorphic in a neighborhood of a point $\hat{\mathbf{x}} = (\hat{x}_1, \ldots, \hat{x}_s)$.

For a non-negative integer array $\mathbf{j} = [j_1, \ldots, j_s]$ write $|\mathbf{j}| = j_1 + j_2 + \cdots + j_s$ and then $\mathbf{x}^{\mathbf{j}} = x_1^{j_1} x_2^{j_2} \ldots x_s^{j_s}$ is a monomial in $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \ldots, x_s]$ of total degree $|\mathbf{j}|$. Let $(\mathbf{x} - \hat{\mathbf{x}})^{\mathbf{i}}$ denote $(x_1 - \hat{x}_1)^{i_1} \ldots (x_s - \hat{x}_s)^{i_s}$ Assume a global degree ordering is given on the monomials of $\mathbb{C}[\mathbf{x}]$. As in [9, 3, 2] we use the differentiation operator

$$\partial_{\mathbf{x}^{\mathbf{j}}} \equiv \partial_{x_1^{j_1} \cdots x_s^{j_s}} \equiv \frac{1}{j_1! \cdots j_s!} \frac{\partial^{j_1 + \cdots + j_s}}{\partial x_1^{j_1} \cdots \partial x_s^{j_s}}. \tag{1}$$

where we write $\partial_{\mathbf{x}^{\mathbf{j}}}[\hat{\mathbf{x}}](f)$ to indicate that we have applied the operator to function f and evaluated at point $\hat{\mathbf{x}}$.

The *local array of degree k at $\hat{\mathbf{x}}$*, $\mathbf{L}(k, \hat{\mathbf{x}})$ is the $n\binom{k+s-1}{s} \times \binom{k+s}{s}$ matrix with columns indexed by the differentials $\partial_{\mathbf{x}^{\mathbf{j}}}$ for $|\mathbf{j}| \leq k$ or, more commonly, just by the $\mathbf{x}^{\mathbf{j}}$ ordered, left to right, by the degree lexicographical ordering. In particular, the left hand column has index 1 for the evaluation functional. The rows will be indexed by the functions $\mathbf{x}^{\mathbf{i}} f_\alpha$ for $|\mathbf{i}| < k$, $\alpha = 1, \ldots, n$. Again these will be grouped by degree $|\mathbf{i}|$ and by monomial $\mathbf{x}^{\mathbf{i}}$ in lexicographical order. In particular the first n rows are indexed by $f_1, \ldots, f_n$.

The entry in the row indexed by $\mathbf{x}^{\mathbf{i}} f_\alpha$ and column indexed by $\mathbf{x}^{\mathbf{j}}$ is $\partial_{\mathbf{x}^{\mathbf{j}}}[\hat{\mathbf{x}}]((\mathbf{x} - \hat{\mathbf{x}})^{\mathbf{i}} f_\alpha)$.

We write nullity (L) for the right null dimension, i.e the dimension of the right nullspace of L. It is the dimension of the right approxi-nullspace of L if we are using numerical linear algebra. This is defined in [6] or one may obtain reasonably similar results by counting the number of columns minus the number of singular values of L greater than the tolerance ε.

The reason for using $\partial_{\mathbf{x}^\mathbf{j}}[\hat{\mathbf{x}}]((\mathbf{x}-\hat{\mathbf{x}})^\mathbf{i} f_\alpha)$ instead of simply $\partial_{\mathbf{x}^\mathbf{j}}[\hat{\mathbf{x}}](\mathbf{x}^\mathbf{i} f_\alpha)$ in the definition of $\mathbf{L}(k,\hat{\mathbf{x}})$, which would still give a row-equivalent matrix, is from

$$\partial_{\mathbf{x}^\mathbf{j}}[\hat{\mathbf{x}}]((\mathbf{x}-\hat{\mathbf{x}})^\mathbf{i} f) = \begin{cases} \partial_{\mathbf{x}^{\mathbf{j}-\mathbf{i}}}[\hat{\mathbf{x}}](f) & \text{if } \mathbf{j} > \mathbf{i} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

one obtains a block structure

$$\mathbf{L}(k,\hat{\mathbf{x}}) = \left[\begin{array}{c|c} \mathbf{L}(k-1,\hat{\mathbf{x}}) & M \\ \hline \mathbf{0} & N \end{array}\right] \tag{3}$$

where M, N have $\binom{k+s-1}{k}$ columns. It then follows that nullity $(\mathbf{L}(k-1,\hat{\mathbf{x}})) \leq$ nullity $(\mathbf{L}(k,\hat{\mathbf{x}}))$ In case $f_1,\ldots,f_n$ are exact polynomials, $\hat{\mathbf{x}}$ is an isolated zero of the system $f_1,\ldots,f_n$, and the nullity is calculated exactly, it is known [3] that if nullity $(\mathbf{L}(k-1,\hat{\mathbf{x}})) =$ nullity $(\mathbf{L}(k,\hat{\mathbf{x}}))$ then

$$m = \text{nullity}\,(\mathbf{L}(k,\hat{\mathbf{x}})) = \text{nullity}\left(\mathbf{L}(k',\hat{\mathbf{x}})\right) \text{ for all } k' \geq k. \tag{4}$$

This common value m is the multiplicity of the system at $\hat{\mathbf{x}}$. Further, we may recover a polynomial system from $\mathbf{L}(k,\hat{\mathbf{x}})$ by multiplying $\mathbf{L}(k,\hat{\mathbf{x}})$ by the column vector of column indices. It is shown in [2, Lemma 3] that these polynomials form a basis for the local ring $\mathcal{O}_{\hat{\mathbf{x}}}$ of $\mathbb{C}[x_1,\ldots,x_s]/\langle f_1,\ldots,f_n\rangle$ at $\hat{\mathbf{x}}$.

In fact the statements of the last paragraph are true also if $f_1,\ldots,f_n$ are holomorphic functions and $\hat{\mathbf{x}}$ is isolated in the sense that the local ring $\mathcal{O}_{\hat{\mathbf{x}}}$ is a finite dimensional $\mathbb{C}$-algebra. Then the argument in [2, §3] holds. In fact, if k is greater than the nil-index (called *depth* in [3]), which implies (4), then the local ring of the holomorphic system $f_1,\ldots,f_n$ is just the local ring of the polynomial system $\tilde{f}_1,\ldots,\tilde{f}_n$ where $\tilde{f}_i$ is the Taylor series of f_i truncated above degree k.

When the nullity is calculated approximately, unless the system is given very accurately and the tolerance is very tight, the equations obtained from the local array may not be consistent and/or may give an ideal with more than one distinct point. The motivation for [2] and the present paper is to examine this situation.

3 Constructing $\mathcal{H}(\mathcal{U} \cap V_F)$ from Differentials

In this section we construct a zero dimensional ring knowing the points of the variety and the local rings at the points. In particular we show how to compute the multiplication matrices of a zero dimensional affine algebra from an arbitrary basis of the space of differential functionals. We can then calculate the zeros and/or a system of polynomial equations defining this algebra. In this section we are working strictly with polynomials, i.e. by affine algebra we mean a quotient algebra of $\mathbb{C}[x_1,\ldots,x_s]$.

Following [9, 3], a *differential functional* on $\mathbb{C}[\mathbf{x}]$ is a formal sum $\mathbf{d} = \sum_{\mathbf{x}^\mathbf{j}} c_{\mathbf{x}^\mathbf{j}} \partial_{\mathbf{x}^\mathbf{j}}[\hat{\mathbf{x}}]$ where $\mathbf{x}^\mathbf{j}$ runs over all monomials in $\mathbb{C}[\mathbf{x}]$, each $c_{\mathbf{x}^\mathbf{j}}$ is a complex number and $\hat{\mathbf{x}}$ denotes a point we will call the *center*, this point will act as a local origin. Although differential functional $\mathbf{d}$ may have infinitely many non-zero terms, when applied to any polynomial $f \in \mathbb{C}[\mathbf{x}]$ and evaluated at $\hat{\mathbf{x}}$ all but finitely many terms of $\mathbf{d}$ vanish on f, so $\mathbf{d}(f)$ is well defined. Note that by allowing

infinite formal sums each functional can be written in terms of a functional evaluated at the origin because of the following change of center formula:

$$\partial_{\mathbf{x}^\mathbf{j}}[\hat{\mathbf{x}}] = \sum_{\mathbf{i}\geq\mathbf{j}} \tbinom{i_1}{j_1}\hat{x}_1^{i_1-j_1}\cdots\tbinom{i_s}{j_s}\hat{x}_s^{i_s-j_s}\partial_{\mathbf{x}^\mathbf{i}}[\hat{\mathbf{o}}] \tag{5}$$

where $\hat{\mathbf{x}} = [\hat{x}_1, \ldots, \hat{x}_s]$, $\mathbf{i} = [i_1, \ldots, i_s]$ and $\mathbf{j} = [j_1, \ldots, j_s]$, $\mathbf{i} \geq \mathbf{j}$ means $i_\alpha \geq j_\alpha$ for all $1 \leq \alpha \leq s$ and, for this formula, $x_i^0 = 1$ even if $x_i = 0$. Further, $\hat{\mathbf{o}}$ always denotes the actual origin in that all coordinates of $\hat{\mathbf{o}}$ are 0.

If $\mathcal{I} = \langle f_1, \ldots, f_n\rangle$ is an ideal of $\mathbb{C}[\mathbf{x}]$ then $\mathbf{d}$ is said to be a *differential functional on* $\mathbb{C}[\mathbf{x}]/\mathcal{I}$, or by abuse of notation *on* $\mathcal{I}$, if $\mathbf{d}(f) = 0$ for all $f \in \mathcal{I}$. By the duality given in [9, §2.3] for a zero dimensional ideal $\mathcal{I}$ the vector space of all differential functionals on $\mathcal{I}$ has the same $\mathbb{C}$-dimension as the $\mathbb{C}$-algebra $\mathbb{C}[\mathbf{x}]/\mathcal{I}$.

Let $\mathcal{D}[\mathcal{I}]$ be the dual space of all differential functionals on zero dimensional ideal $\mathcal{I}$ and $\mathcal{D} = \mathbf{d}_1, \ldots, \mathbf{d}_m$ be a $\mathbb{C}$-basis for $\mathcal{D}[\mathcal{I}]$. The *dual array* ${}^{\mathcal{D}}D_k$ of degree k will be the $\binom{k+s}{k} \times m$ matrix with rows indexed by monomials of total degree $\leq k$ in the same ordering as the local matrix of the following section. The entry of ${}^{\mathcal{D}}D_k$ in the row indexed by $\mathbf{x}^\mathbf{i}$ and column j is just $\mathbf{d}_j(\mathbf{x}^\mathbf{i})$. We write $\mathbf{r}_{\mathbf{x}^\mathbf{i}}$ for the row of ${}^{\mathcal{D}}D_k$ indexed by $\mathbf{x}^\mathbf{i}$. The duality between $\mathcal{D}[\mathcal{I}]$ and $\mathbb{C}[\mathbf{x}]/\mathcal{I}$ for zero dimensional ideals implies the following.

Lemma 1 *Let $\mathcal{I} = \langle f_1, \ldots, f_n\rangle$ be a zero-dimensional ideal in $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \ldots, x_s]$ with $A = \mathbb{C}[\mathbf{x}]/\mathcal{I}$ of $\mathbb{C}$-dimension m. Suppose $\mathbf{b}_1, \ldots, \mathbf{b}_m$ are monomials in $\mathbb{C}[\mathbf{x}]$. Let $\mathcal{D} = [\mathbf{d}_1, \ldots \mathbf{d}_m]$ be a $\mathbb{C}$-basis for the dual space $\mathcal{D}[\mathcal{I}]$ and ${}^{\mathcal{D}}D_k$ be the dual array of degree k where $k > \deg \mathbf{b}_i$ for all $1 \leq i \leq m$. Then $\mathbf{b}_1, \ldots, \mathbf{b}_m$ is a $\mathbb{C}$-basis for A if and only if $\mathbf{r}_{\mathbf{b}_1}, \ldots, \mathbf{r}_{\mathbf{b}_m}$ is a basis for the row space of ${}^{\mathcal{D}}D_k$.*

Proof: This follows from the fact that $c_1\mathbf{r}_{\mathbf{b}_1} + \ldots c_m\mathbf{r}_{\mathbf{b}_m} = 0$ if and only if $\mathbf{d}_j(c_1\mathbf{b}_1 + \cdots + c_m\mathbf{b}_m) = 0$ for all $1 \leq j \leq m$ if and only if $c_1\mathbf{b}_1 + \cdots + c_m\mathbf{b}_m = 0$ in A. ∎

For a differential functional $\mathbf{d} = \sum_{\mathbf{x}^\mathbf{j}} c_{\mathbf{x}^\mathbf{j}}\partial_{\mathbf{x}^\mathbf{j}}[\hat{\mathbf{o}}]$ and $k \geq 0$ we will write $\mathbf{d}^{(k)} = \sum_{|\mathbf{j}|\leq k} c_{\mathbf{x}^\mathbf{j}}\partial_{\mathbf{x}^\mathbf{j}}[\hat{\mathbf{o}}]$. The corresponding vector in the column space of ${}^{\mathcal{D}}D_k$ with entries $c_{\mathbf{x}^\mathbf{j}}, |\mathbf{j}| \leq k$ will be denoted $\tilde{\mathbf{d}}^{(k)}$. Conversely if a vector $\tilde{\mathbf{d}}^{(k)}$ is given then the differential $\mathbf{d}^{(k)}$ is defined as above even if $\mathbf{d}$ has not been defined. Finally, if we are working around a general $\hat{\mathbf{x}}$ we can distinguish between the vector with center $\hat{\mathbf{x}}$ and its translate about $\hat{\mathbf{o}}$ by notation $\tilde{\mathbf{d}}^{(k)}[\hat{\mathbf{x}}]$ and $\tilde{\mathbf{d}}^{(k)}[\hat{\mathbf{o}}]$.

Assume, with notation as in the lemma, that $\mathcal{B} = [\mathbf{b}_1, \ldots, \mathbf{b}_m]$ is a basis for A. Let R be the $m \times m$ matrix with rows $\mathbf{r}_{\mathbf{b}_1}, \ldots, \mathbf{r}_{\mathbf{b}_m}$. By Lemma 1 R is invertible so let $D_k^\mathcal{B} = {}^{\mathcal{D}}D_kR^{-1}$. So $D_k^\mathcal{B}$ is also dual array for $\mathcal{D}[\mathcal{I}]$ but now satisfies

$$\mathbf{d}_j^{(k)}(\mathbf{b}_i) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \tag{6}$$

where $\tilde{\mathbf{d}}_1^{(k)}, \ldots, \tilde{\mathbf{d}}_m^{(k)}$ are the columns of $D_k^\mathcal{B}$. We will call $D_k^\mathcal{B}$ the *dual array* associated with the basis $\mathcal{B} = [\mathbf{b}_1, \ldots, \mathbf{b}_m]$.

The *multiplication matrix* $M_a^{\mathcal{B}}$ of a finite $\mathbb{C}$-algebra $A = \mathbb{C}[\mathbf{x}]/\mathcal{I}$, $a \in \mathbb{C}[\mathbf{x}]$ with respect to $\mathbb{C}$-basis $\mathcal{B} = [\mathbf{b}_1, \ldots, \mathbf{b}_m]$ of A is the $m \times m$ matrix such that if $f = \alpha_1\mathbf{b}_1 + \cdots + \alpha_m\mathbf{b}_m$

$$[\alpha_1\ \alpha_2\ \ldots\ \alpha_m]M_a^{\mathcal{B}} = [\beta_1\ \ldots\ \beta_m] \text{ where } af = \beta_1\mathbf{b}_1 + \cdots + \beta_m\mathbf{b}_m. \tag{7}$$

In particular the i^{th} row of M_a is $[\gamma_1, \ldots, \gamma_m]$ where $a\mathbf{b}_i = \gamma_1\mathbf{b}_1 + \ldots \gamma_1\mathbf{b}_m$.

Now if $\mathbf{a}$ is a monomial and $\mathbf{d}_1^{(k)}, \ldots, \mathbf{d}_m^{(k)}$ are the differentials associated with $D_k^{\mathcal{B}}$ then by definition $\mathbf{d}_j^{(k)}(\mathbf{ab}_i)$ is the entry in the j^{th} column of the row indexed by $\mathbf{ab}_i$. On the other hand, if $\mathbf{ab}_i = \gamma_1\mathbf{b}_1 + \ldots \gamma_1\mathbf{b}_m$ as above, $\mathbf{d}_j^{(k)}(\mathbf{ab}_i) = \gamma_j$ by (6).

This proves

Lemma 2 *Let $\mathbf{b}_1, \ldots, \mathbf{b}_m$ be a $\mathbb{C}$-basis for $A = \mathbb{C}[\mathbf{x}]/\mathcal{I}$, $\mathbf{a}$ a monomial in $\mathbb{C}[\mathbf{x}]$. Let $k \geq \deg \mathbf{ab}_i$ for $1 \leq i \leq m$. Let $\mathbf{r}_{\mathbf{ab}_i}$ be the row corresponding to $\mathbf{ab}_i$ of the dual matrix $D_k^{\mathcal{B}}$. Then the multiplication matrix is given by*

$$M_{\mathbf{a}}^{\mathcal{B}} = \begin{bmatrix} \mathbf{r}_{\mathbf{ab}_1} \\ \vdots \\ \mathbf{r}_{\mathbf{ab}_m} \end{bmatrix}.$$

In this paper we may not be starting with the known differential functionals on the ideal of a given ring. In fact our main goal is to be able to construct a ring with given functionals. Not every matrix of the correct size represents ${}^{\mathcal{D}}D_k$ of some dual basis of an ideal. More generally, not every set of differential functionals defined on $\mathbb{C}[\mathbf{x}]$ defines the dual space of an ideal. The necessary and sufficient condition is given in [9, Theorem 2.21] and subsequent remarks there. This conditon can be summarized as follows. For a linear functional $\mathbf{d} : \mathbb{C}[\mathbf{x}] \to \mathbb{C}$ write $\mathbf{d} \cdot \mathbf{x}^{\mathbf{j}}$ for the functional

$$\mathbf{d} \cdot \mathbf{x}^{\mathbf{j}}(f) = d(\mathbf{x}^{\mathbf{j}} f)$$

A vector space $\mathbf{D}$ of linear functionals is *closed* if whenever $\mathbf{d} \in \mathbf{D}$ then also $\mathbf{d} \cdot \mathbf{x}^{\mathbf{j}} \in \mathbf{D}$ for all $\mathbf{j}$. Actually it is enough to check this condition for the linear monomials x_j. Then a finite dimensional vector space of linear functionals $\mathbf{D}$ is the space of differential funtionals for some ideal $\mathcal{I}$ if and only if $\mathbf{D}$ is closed.

One can try to check this condition on the spaces spanned by matrices in the form of ${}^{\mathcal{D}}D_k$ or $D_k^{\mathcal{B}}$. An easy necessary condition for closedness is that the multiplication matrices as constructed in Lemma 2 must satisfy $M_{x_i}^{\mathcal{B}} M_{x_j}^{\mathcal{B}} = M_{x_j}^{\mathcal{B}} M_{x_i}^{\mathcal{B}}$ for all $1 \leq i < j \leq s$. In general if the differential functional $\mathbf{d} = \sum_{\mathbf{x}^{\mathbf{j}}} c_{\mathbf{x}^{\mathbf{j}}} \partial_{\mathbf{x}^{\mathbf{j}}}[\hat{\mathbf{o}}]$ has infinitely many non-zero terms one can not computationally determine if the $\mathbf{d} \cdot x_j \in \mathbf{D}$, although by trying a small part, i.e. working in some ${}^{\mathcal{D}}D_k$, one can show a given system is not closed. If one is working locally, then the differential functionals have only finitely many non-zero terms and a definitive computational test is possible.

When $a = x_i$, treating $\mathcal{B}$ as a column vector, we can write equation (7) heuristically in the form

$$M_{x_i}^{\mathcal{B}} \mathcal{B} = x_i \mathcal{B} \tag{8}$$

This suggests that if x_i and $\mathcal{B}$ are evaluated at a zero of the system, i.e. a point on the associated variety, then x_i is an eigenvalue of M_{x_i} and $\mathcal{B}$ is an eigenvector. In the absence of multiple

zeros this is exactly the Möller-Stetter solution method. In all cases equation (8) is a system of equations in the ideal of the system (note [9, Exercises §2.2]). Since the multiplication matrices give a faithful representation of the affine ring $\mathbb{C}[\mathbf{x}]/\mathcal{I}$ and these equations define the multiplication matrices it follows that these equations, running over all x_i, form a basis for the ideal $\mathcal{I}$. Thus we have shown

Theorem 1 *Let* $\mathbf{D}$ *be a closed finite dimensional vector space of linear functionals* $\mathbb{C}[\mathbf{x}] \to \mathbb{C}$. *Then the system of equations* (8) *defines the the ideal dual to* $\mathbf{D}$.

We will apply this in the following manner.

Corollary 1 *Given finitely many zero dimensional local rings* $\mathcal{O}_{\hat{\mathbf{x}}_i}$ *at distinct points* $\hat{\mathbf{x}}_i \in \mathbb{C}^s, 1 \leq i \leq \ell$, *the method above will construct a defining system of equations for an ideal* $\mathcal{I} \subset \mathbb{C}[x_1, \ldots, x_s]$ *with exactly the given local rings.*

Proof: Applying the change of center formula (5) to the closed set of functionals centered at $\hat{\mathbf{x}}_i$ defining $\mathcal{O}_{\hat{\mathbf{x}}_i}$ gives a closed set of functionals $\mathbf{D}_i$ at $\hat{\mathbf{o}}$. Moreover $\mathbf{D}_1 \oplus \cdots \oplus \mathbf{D}_\ell$ is then a closed finite dimensional space of functionals so Theorem 1 applies.

Before looking at several examples we note that equations (8) are subequations, via Lemma 2 of the equation

$$\mathbf{X}_k - D_k^{\mathcal{B}}\mathcal{B} \tag{9}$$

where $\mathbf{X}_k$ is a column vector of the monomials of degree $\leq k$ in the ordering used in §2, typically the degree lexicographical order. See (10) below.

Example 1: Condsider the first system given by Stetter in [9, Example 2.13], here $\hat{\mathbf{z}}_1 = (2, 1)$ and $\hat{\mathbf{z}}_2 = (1, 2)$. Change of center (5) is a linear transformation on ${}^{\mathcal{D}}D_3$ so changing from $\hat{\mathbf{z}}_1, \hat{\mathbf{z}}_2$ to $\hat{\mathbf{o}} = (0, 0)$ is accomplished by the unit diagonal lower triangular matrices

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 8 & 12 & 0 & 6 & 0 & 0 & 1 & 0 & 0 & 0 \\ 4 & 4 & 4 & 1 & 4 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 4 & 0 & 2 & 2 & 0 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & 0 & 3 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 3 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 4 & 1 & 2 & 2 & 0 & 0 & 1 & 0 & 0 \\ 4 & 4 & 4 & 0 & 4 & 1 & 0 & 0 & 1 & 0 \\ 8 & 0 & 12 & 0 & 0 & 6 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus the differentials $\mathcal{D} = [d_1, d_2, d_3, d_4]$ given by Stetter, assume variables x, y,

$$\begin{array}{llll} d_1 = & 2\partial_1[\hat{\mathbf{z}}_1] & -3\partial_x[\hat{\mathbf{z}}_1] & +5\partial_{x^2}[\hat{\mathbf{z}}_1] \quad -\partial_1[\hat{\mathbf{z}}_2] \\ d_2 = & & +\partial_x[\hat{\mathbf{z}}_1] & -4\partial_{x^2}[\hat{\mathbf{z}}_1] \\ d_3 = & -\partial_1[\hat{\mathbf{z}}_1] & +\partial_x[\hat{\mathbf{z}}_1] & -\partial_{x^2}[\hat{\mathbf{z}}_1] \quad +\partial_1[\hat{\mathbf{z}}_2] \\ d_4 = & & & \partial_{x^2}[\hat{\mathbf{z}}_1] \end{array}$$

give rise to the matrix ${}^{\mathcal{D}}D_3$ at $\hat{\mathbf{o}}$. Since the first four rows are the 4×4 identitiy then this matrix is already $D_3^{\mathcal{B}}$ for $\mathcal{B} = [1, x, y, x^2]$. So we may find the generators fo the ideal of this system and the multiplication matrices from (9).

$$\mathbf{X}_3 - D_3^{\mathcal{B}}\mathcal{B} = \begin{bmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \\ x^3 \\ x^2y \\ xy^2 \\ y^3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 1 & 0 \\ -2 & 0 & 3 & 0 \\ 9 & -12 & -1 & 6 \\ -1 & 0 & 1 & 1 \\ -3 & 1 & 3 & 0 \\ -6 & 0 & 7 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \\ x^2 \end{bmatrix} \tag{10}$$

The multiplication matrices are then given by

$$M_x = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 1 & 0 \\ 9 & -12 & -1 & 6 \end{bmatrix}, \quad M_y = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 \\ -2 & 0 & 3 & 0 \\ -1 & 0 & 1 & 1 \end{bmatrix}$$

Compare with [9, Ex. 2.9].

We get directly from (10) the generators $xy+1-x-y, x^3-9+12x+y-6x^2, y^2+2-3y, x^2y+1-y-x^2$ of the ideal $\mathcal{I}$ of Stetter's ring. It is not hard to see that the last generator above is redundant. A Gröbner basis calculation shows that none of the first three generators are redundant.

From these generators we can now use the method of [3] reviewed in the previous section to find differentials defining the local ring at $\hat{\mathbf{z}}_1$ and $\hat{\mathbf{z}}_2$. At $\hat{\mathbf{z}}_1$ the differentials are $1, \partial_x[\hat{\mathbf{z}}_1], \partial_{x^2}[\hat{\mathbf{z}}_1]$ while at the simple zero $\hat{\mathbf{z}}_2$ there is just $\partial_1[\hat{\mathbf{z}}_2]$. We can now construct the $\mathcal{H}(\mathcal{U} \cap V_F)$ with these two points using Corollary 1 by the dual array ${}^{\mathcal{D}'}D_3$ consisting of the first three columns of C_1 above and the first column of C_2. The 4×4 submatrix of the first 4 rows of ${}^{\mathcal{D}'}D_3$ is

$$R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 4 & 0 & 1 & 2 \\ 1 & 4 & 0 & 1 \end{bmatrix}$$

which is easily seen to be invertible. It should come as no suprise that ${}^{\mathcal{D}'}D_3 R = D_3^{\mathcal{B}}$ above as the ideal of Stetter's example only has only these two closed points. ∎

Example 2: Consider the system

$$F = \begin{bmatrix} x^2\cos(5x) + x\sin(10y) - x \\ y^2\sin(5x) + y\cos(10y) - y \end{bmatrix} \tag{11}$$

Using, perhaps, the method of Example 3 below, one finds a zero of multiplicity 3 at $\hat{\mathbf{o}} = (0, 0)$, a simple zero at $\hat{\mathbf{y}} = (0.22857 + 0.26856\imath, 0.03744 + 0.01531\imath)$ and a double zero at $\hat{\mathbf{z}} = (0.20820 + 0.33027\imath, 0)$, all numbers rounded to 5 digits for display but calculated by

default precision in MAPLE. Calculating the local matrix at the origin with a tight tolerance one gets differential functionals $\partial_1[\hat{\mathbf{o}}], \partial_y[\hat{\mathbf{o}}]$ and $\partial_{y^2}[\hat{\mathbf{o}}]$ on the local ideal at $\hat{\mathbf{o}}$. At simple zero $\hat{\mathbf{y}}$ there is only the differential $\partial_1[\hat{\mathbf{y}}]$ and at $\hat{\mathbf{z}}$ there are the two differentials $\partial_1[\hat{\mathbf{z}}], \partial_x[\hat{\mathbf{z}}]$.

As in Example 1 we can change the center of these differentials to $\hat{\mathbf{o}}$ and we see that the first 6 rows of ${}^{\mathcal{D}}D_k, k \geq 2$ form a well conditioned matrix so $\mathcal{B} = [1, x, y, x^2, xy, y^2]$ can be our $\mathbb{C}$-basis. Since all these monomials are quadratic it is sufficient to work with $D_3^{\mathcal{B}}$. Equation (9) is then

$$\begin{bmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \\ x^3 \\ x^2y \\ xy^2 \\ y^3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -.05607 - .05537\imath & .000 & .40478 + .28440\imath & .50034 - .08629\imath & .000 \\ 0 & .04575 - .07918\imath & .000 & .23407 + .00902\imath & .24019 + .64469\imath & .000 \\ 0 & .05115 - .02499\imath & .000 & -.01570 + .14450\imath & -.19663 + .00629\imath & .000 \\ 0 & .00389 - .00524\imath & .000 & .00604 + .01559\imath & -.01970 + .01100\imath & .000 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \end{bmatrix}$$

Here 0 means 0 but .000 indicates a complex number of absolute value less than .001 .

Eliminating trivial equations we get the following 4 generators for the ideal of $\mathcal{H}(\mathcal{U} \cap V_F)$:

$$\begin{aligned} g_1 =& x^3 - (-.05607 - .05537\imath)x - (.40478 + .28440\imath)x^2 - (.50034 - .08629\imath)xy \\ g_2 =& x^2y - (.04575 - .07918\imath)x - (.23407 + .00902\imath)x^2 - (.24019 + .64469\imath)xy \\ g_3 =& xy^2 - (.05115 - .02499\imath)x - (-.01570 + .14450\imath)x^2 - (-.19663 + .00629\imath)xy \\ g_4 =& y^3 - (.00389 - .00524\imath)x - (.00604 + .01559\imath)x^2 - (-.01970 + .01100\imath)xy \end{aligned}$$

This system has only three distinct zeros: $\hat{\mathbf{o}}$ of multipicity 3, $\hat{\mathbf{y}}$ of multiplicity 1 and $\hat{\mathbf{z}}$ of multiplicity 2. An independent calculation of the local arrays of this system at these points will confirm the same local structure, i.e. essentially the same differential functionals as the original system F. Thus these polynomials define the ideal of $\mathcal{H}(\mathcal{U} \cap V_F)$. We will see in Example 3 below the relationship between this system and one derived directly from a local matrix of F. It is possible that one or more of these equations is still redundant, experiments with PHCPACK and BERTINI [1, 11] suggest that g_2 may not be needed, but g_4 is independent of the other three equations and is necessary. Good software for numerical Gröbner bases would have been helpful here.

4 Approximate Differential Functionals and Singular Values of the Local Array

In [2] we took the position that a k satisfying equation (4) was sufficiently large not only for theoretical use, but also for numerical use. In this paper we see that larger k do give more accurate results as shown in the following claim. This claim asserts that the actual differential

functionals on ideal $\mathcal{I} = \langle f_1, \ldots, f_s \rangle$ acting on $\mathbf{L}(k, \hat{\mathbf{x}})$ will have arbitrarily small residues provided k is large enough. We assume without loss of generality that the center is the origin $\hat{\mathbf{x}} = \hat{\mathbf{o}}$.

Claim 1 *Let $f_1, \ldots, f_n$ be a holomorphic system at the origin $\hat{\mathbf{o}}$ in variables $x_1, \ldots, x_s$. Let r, $0 < r < 1$, be less than the minimum radius of convergence of a convergent series represention of each f_i. Let $\hat{\mathbf{y}}$ be a zero with $\|\hat{\mathbf{y}}\| < r$ of multiplicity m with differentials $\mathbf{d}_1, \ldots, \mathbf{d}_m$ evaluated at $\hat{\mathbf{y}}$, i.e. vectors $\tilde{\mathbf{d}}_1^{(k)}[\hat{\mathbf{y}}], \ldots, \tilde{\mathbf{d}}_m^{(k)}[\hat{\mathbf{y}}]$ are in the null space of $\mathbf{L}(k, \hat{\mathbf{y}})$ for sufficiently large k. Then for any $\varepsilon > 0$ there is an integer K so that if $k \geq K$ and $1 \leq j \leq m$ then $\|\mathbf{L}(k, \hat{\mathbf{o}})\tilde{\mathbf{d}}_j^{(k)}[\hat{\mathbf{o}}]\| < \varepsilon$.*

I do not have a clean proof of this at present. This statement does seem easier for a polynomial system as then there are only finitely many terms and the coefficients are bounded, which might not occur for analytic functions. Moreover the matrix $\mathbf{L}(k, \hat{\mathbf{o}})$ is sort of a band matrix. An argument for the polynomial case might go as follows. In the change of center formula (5) each coefficient is a product of factors $\binom{i}{j} y^{i-j} = \binom{j+\alpha}{j} y^{\alpha}$ for $\alpha = i - j$. But $\sum_{\alpha=0}^{\infty} \binom{j+\alpha}{j} y^{\alpha} = \frac{1}{(1-y)^{j+1}}$ is an absolutely convergent series for $j > 0$ and $|y| < 1$. Hence the lower entries of $\tilde{\mathbf{d}}_j^{(k)}$ become small as k gets large. Meanwhile the entries in $\mathbf{L}(k, \hat{\mathbf{o}})$ remain bounded in both absolute value and number of non-zero entries in a row. If the highest total degree of any f_i is p then the property of $\mathbf{d}_j$ being a differential gives a zero in the first $\binom{k-p+s}{s}$ positions of $\mathbf{L}(k, \hat{\mathbf{o}})\tilde{\mathbf{d}}_j^{(k)}$, the later terms being a constant times a tail of the absolutely convergent series of coefficients of $\tilde{\mathbf{d}}_j$.

In the claim note that the restriction $\|\hat{\mathbf{y}}\| < r < 1$ is somewhat artificial, although apparently needed for the argument, see Example 1. However the system can be rescaled so that any particular zero $\hat{\mathbf{y}}$ in the region of series convergence satisfies this restriction. The cost, however, will generally be that K will have to be much larger.

Motivated by [6] we take the following point of view on singular values. Let A be a complex $p \times q$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_q \geq 0$. Then there exist unit vectors $\mathbf{v}_1, \ldots, \mathbf{v}_q \in \mathbb{C}^q$ so that $\sigma_q = \|A\mathbf{v}_q\| = \min\{\|A\mathbf{v}\| \,\big|\, \mathbf{v} \in \mathbf{V}_q, \|\mathbf{v}\| = 1\}$ where $V_q = \mathbb{C}^q$, $\sigma_{q-1} = \|A\mathbf{v}_{q-1}\| = \min\{\|A\mathbf{v}\| \,\big|\, \mathbf{v} \in \mathbf{V}_{q-1}, \|\mathbf{v}\| = 1\}$ and so on where V_{q-j} is the orthogonal complement of the subspace of $\mathbb{C}^q$ spanned by $v_q, v_{q-1}, \ldots, v_{q-j+1}$. The v_{q-j} are not uniquely defined but the σ_{q-j} are, in fact if $A = USV^{\top}$ is a singular value decomposition of A then $\mathbf{v}_{q-j}$ can be the $q - j^{\text{th}}$ column of V. Note transposes in this context are Hermitian transposes and norms are 2-norms.

Thus if $\mathbf{L}(k, \hat{\mathbf{x}})$ has m singular values smaller than ε then there are m orthogonal vectors $\tilde{\mathbf{d}}_1^{(k)}, \ldots, \tilde{\mathbf{d}}_m^{(k)}$ with $|\mathbf{L}(k, \hat{\mathbf{x}})\tilde{\mathbf{d}}_j^{(k)}| < \varepsilon$. For small ε these approximately define differential functionals on the $\mathbb{C}$-linear subspace of $\mathcal{I}$ generated by polynomials $(\mathbf{x} - \hat{\mathbf{x}})^{\mathbf{i}} f_{\alpha}$. As k gets large this $\mathbb{C}$-linear subspace, which is an approxi-nullspace of $\mathbf{L}(k, \hat{\mathbf{o}})$, includes most of $\mathcal{I}$ so these $\mathbf{d}_i^{(k)}$ are approximately differential functionals on $\mathcal{I}$. By Lemma 1 we should eventually discover all differentials of points near $\hat{\mathbf{x}}$, but since the converse of Claim 1 is not true we may obtain some $\tilde{\mathbf{d}}$ which are not actual differential functionals on the ideal of the system.

We also must mention that although the ε-approxi-nullspace of $\mathbf{L}(k, \hat{\mathbf{o}})$ contains vectors $\tilde{\mathbf{d}}^{(k)}$ with $|\mathbf{L}(k, \hat{\mathbf{x}})\tilde{\mathbf{d}}^{(k)}| < \varepsilon\|\tilde{\mathbf{d}}^{(k)}\|$, the converse of this is also not true. Thus if $\mathbf{d}[\hat{\mathbf{y}}]$ is a differential functional on $\mathcal{I}$, $\tilde{\mathbf{d}}^{(k)}[\hat{\mathbf{o}}]$ may not be in an approxi-nullspace of $\mathbf{L}(k, \hat{\mathbf{o}})$. In practice, however, $\tilde{\mathbf{d}}^{(k)}[\hat{\mathbf{o}}]$ is close to the approxi-nullspace for large k.

Example 3: We again consider the system (11) of Example 2. In this case we assume we do not already know the local structure and use the results of this section to find it. Let $\hat{\mathbf{x}} = (.09368 + .16038\imath, -.00308 - .00813\imath)$, a point chosen randomly in a neighborhood containing the zeros of Example 2. Again, actual computations were done in default precision in MAPLE, but are reported here to 5 digit precision. Note that, unlike in [2], $\hat{\mathbf{x}}$ is not an approximate zero of the system. We calculate the local array $\mathbf{L}(13, \hat{\mathbf{x}})$, a 182×105 matrix. The smallest 7 singular values are .26e-4, .17e-5, .18e-7, .52e-8, .20e-9, .13e-10, .69e-13, thus it is only with some hindsight that we find the approxi-nullspace with tolerance .5e-5 which gives us a nullspace of dimension 6.

The 6×6 matrix of the top 6 rows of the dual array ${}^{\mathcal{D}}D_{13}$ is nicely conditioned thanks to the fact that our software returned an orthonormal basis for the approxi-nullspace. Therefore we can again use the $\mathbb{C}$-basis $\mathcal{B} = [1, x, y, x^2, xy, y^2]$. Since $x\mathcal{B}, y\mathcal{B}$ have terms of degree 3 or less we only need to use ${}^{\mathcal{D}}D_3$, i.e. the first 10 rows of ${}^{\mathcal{D}}D_{13}$. At this point we could proceed with this matrix and calculate $D_3^{\mathcal{B}}$ and then directly calulate the multiplication matrices and then the zeros and/or the equations of the *numerical local ring*, being careful to remember that *the origin has been changed to* $\hat{\mathbf{x}}$. Thus a change of variables would be required to compare the results with Example 1 which uses the natural origin $\hat{\mathbf{o}}$.

Instead, to directly compare with Example 2, we apply the change of center formula (5) to the functionals approximated by the columns of ${}^{\mathcal{D}}D_{13}$ to change the base point to $\hat{\mathbf{o}}$. This should work since the equations in (11) are entire and $\|\hat{\mathbf{x}} - \hat{\mathbf{o}}\| < 1$. We note that (5) gives a linear transformation between the functionals at $\hat{\mathbf{x}}$ with those at $\hat{\mathbf{o}}$, so we can perform the change via a matrix multiplication of the array ${}^{\mathcal{D}}D_{13}$. In fact, this change of base point matrix is lower triangular unit diagonal so we can work directly with ${}^{\mathcal{D}}D_3$. It is no suprise that the top 6×6 matrix is still well conditioned so $\mathcal{B}$ still works as a $\mathbb{C}$-basis and we obtain a dual array $D_3^{\mathcal{B}}$ centered at $\hat{\mathbf{o}}$. We now compare with the corresponding array in Example 2, it is sufficient to give the last 4 rows since the first 6 are identical.

Current example last 4 rows of $D_3^{\mathcal{B}}$:

$$\begin{bmatrix} .000, & -.05764 - .05400\imath & .000 & .40393 + .27918\imath & .50882 - .08873\imath & .000 \\ .000 & .04507 - .07803\imath & .000 & .23067 + .00892\imath & .23968 + .63905\imath & .000 \\ .000 & .05020 - .02445\imath & .000 & -.01559 + .14220\imath & -.19204 + .00594\imath & .000 \\ .000 & .00383 - .00514\imath & .000 & .00589 + .01531\imath & -.01926 + .01071\imath & .000 \end{bmatrix}$$

Example 2 last 4 rows of $D_3^{\mathcal{B}}$:

$$\begin{bmatrix} 0 & -.05607 - .05537\imath & .000 & .40478 + .28440\imath & .50034 - .08629\imath & .000 \\ 0 & .04575 - .07918\imath & .000 & .23407 + .00902\imath & .24019 + .64469\imath & .000 \\ 0 & .05115 - .02499\imath & .000 & -.01570 + .14450\imath & -.19663 + .00629\imath & .000 \\ 0 & .00389 - .00524\imath & .000 & .00604 + .01559\imath & -.01970 + .01100\imath & .000 \end{bmatrix}$$

where again .000 indicates a complex number of absolute value less than .001.

The calculation of the multiplication matrices $M_x^{\mathcal{B}}, M_y^{\mathcal{B}}$ is the same as in Example 2. Using the Möller-Stetter method [9] using the refinement method in [2] we obtain the following zeros (remembering to translate by adding $\hat{\mathbf{x}}$).

$$\begin{aligned}
\hat{\mathbf{o}}_1 =&(-.353\text{e-}5 + .332\text{e-}5\imath, .00305 - .00389\imath)\\
\hat{\mathbf{o}}_2 =&(-.365\text{e-}5 + .160\text{e-}5\imath, -.00469 - .00025\imath)\\
\hat{\mathbf{o}}_3 =&(-.372\text{e-}5 + .197\text{e-}5\imath, .00254 - .00397\imath)\\
\hat{\mathbf{y}}_1 =&(.22869 + .26834\imath, .03755 + .01528\imath)\\
\hat{\mathbf{z}}_1 =&(.21401 + .32200\imath, .00539 + .00559\imath)\\
\hat{\mathbf{z}}_2 =&(.19871 + .33279\imath, -.00067 - .00726\imath)
\end{aligned}$$

Newton refinement from the original system F gives $\hat{\mathbf{o}}$ from $\hat{\mathbf{o}}_1, \hat{\mathbf{o}}_2, \hat{\mathbf{o}}_3$ also $\hat{\mathbf{y}}$ from close by $\hat{\mathbf{y}}_1$ and $\hat{\mathbf{z}}$ from $\hat{\mathbf{z}}_1, \hat{\mathbf{z}}_2$ where $\hat{\mathbf{o}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ are as in Example 1.

One can get equations from $M_x^{\mathcal{B}}, M_y^{\mathcal{B}}$ or directly from $D_3^{\mathcal{B}}$ as in Example 2. Although they are close to the equations from Example 2 the ones here are inconsistent and do not give correct multiplicities or local rings. Thus we say that the system of multiplication matrices $M_x^{\mathcal{B}}, M_y^{\mathcal{B}}$ together with designated basis $\mathcal{B}$ forms a *numerical local ring*. This is only a ring in a generalized sense similar to [9, Definition 2.23]. To get an actual ring one should proceed as in Example 2 using the accurate calculation of the zeros by Newton refinement above.

It must be emphasized that the results in this example are highly dependent on choice of center $\hat{\mathbf{x}}$ and choice of how many small singular values to use.

5 The NLR method

Example 3 suggests the following method for finding zeros of an analytic system near $\hat{\mathbf{x}}$ and/or finding the equation of the *numerical local ring* which gives these zeros with their multiplicity structure. While individual parts can be automated, in this paper we will not attempt to automate the method as a whole.

NLR method
Given: $F = [f_1, \ldots, f_n]^\top$ holomorpic system in neighborhood of $\hat{\mathbf{x}} \in \mathbb{C}^s$

- Inspect singular values of $\mathbf{L}(k, \hat{\mathbf{x}})$ for small k, let m be number of small singular values. Find approxi-nullspace N of $\mathbf{L}(k, \hat{\mathbf{x}})$ of dimension m, given as dual array D_k. Find m numerically independent rows of D_k and calculate $D_k^{\mathcal{B}}$ where $\mathcal{B}$ is monomial basis corresponding to these rows.
- Increase k until m singular values get small enough, entries of $D_k^{\mathcal{B}}$ converge or further increase of k is computationally infeasible. If necessary revise $\mathcal{B}$.
- Construct multiplication matrices and calculate m zeros using the method in [2]. Use Newton refinement to see if these are close to true zeros of the system F.

Example 4: The original intended application of the NLR method was to resolve questions on whether the method in [3] was identifying a true multiple zero or perhaps a cluster. The

following example, also given in [2], illustrates this well. Consider the Griewank-Osborne system $G = [g_1, g_2, g_3]^\top$ given in their 1983 paper [4] where

$$\begin{aligned} g_1 &= 0.5x^2\cos(10x) + x\sin(10y) + 10x\sinh(z) \\ g_2 &= \cos(5x) + \sin(5y) - 1 \\ g_3 &= \cosh(5x) + 5\sinh(z) - 1 \end{aligned}$$

An interesting cluster around $\hat{\mathbf{o}}$ is discussed in [2], but we focus on the zeros near $(0, \frac{\pi}{5}, 0)$. We assume this zero is given approximately by $\hat{\mathbf{x}} = (0, .62832, 0)$ and analyze according to the NLR method. We first calculate $\mathbf{L}(5, \hat{\mathbf{x}})$. The smallest 4 singular values are .28, .79e-7, .17e-14, .25e-20. Thus we might conclude as in [3] that we have a zero of multiplicity 3 with tolerance 10^{-7}. This would not be wrong, but rather consistent with the philosophy on numerical multiple points as clusters enunciated originally in [5].

On the other hand calculating the approxi-nullspace with tolerance 1e-7 we find that the first 3 rows of a matrix ${}^{\mathcal{D}}D_k$ give a reasonably well conditioned matrix so we pick $\mathbb{C}$-basis $\mathcal{B} = [1, x, y]$ and calculate $D_2^{\mathcal{B}}$ and then the multiplication matrices which are approximately

$$Mx = \begin{bmatrix} 0 & 1 & 0 \\ .00000 & .00000 & -.4000824 \\ .00000 & .00000 & .0099505 \end{bmatrix},$$

$$My = \begin{bmatrix} 0 & 0 & 1 \\ .00000 & .00000 & .0099505 \\ .00000 & .00000 & -.0002504 \end{bmatrix}, \quad Mz = \begin{bmatrix} .00000 & .00000 & 1.0004123 \\ .00000 & .00000 & .0099546 \\ .00000 & .00000 & -.0002490 \end{bmatrix}$$

where .00000 indicates a real number of absolute value less than .00001 . The fact that these matrices each have an eigenvalue which is not approximately zero is enough to show that we do not have the local ring of a single point. Applying the refinement method of [2] we calculate two zeros at approximately $(0, 0, 0)$, and one $\hat{\mathbf{y}} = (.0099505, -.0002489, -.0002476)$. Recall that this method changes the origin to $\hat{\mathbf{x}}$ so this says that $\hat{\mathbf{x}}$ should be considered a zero of multiplicity 2 with $\hat{\mathbf{y}} + \hat{\mathbf{x}} = (.0099505, .6280711, -.0002476)$ a simple nearby zero. ∎

Example 5: Consider the system

$$\begin{aligned} f_1 &= \sin(x - y) - .5\cos(4y) + .5 \\ f_2 &= \sin(x + y) + 2\cos(5y) - 2. \end{aligned}$$

We first calculate $\mathbf{L}(4, \hat{\mathbf{o}})$ which is a 20×15 matrix. The smallest 5 singular values are approximately $0, .0008, .02, .20, 2.1$ so we take $m = 4$. We calculate ${}^{\mathcal{D}}D_4$ via Li-Zeng with tolerance $\varepsilon = .3$. The condition number of 4×4 matrix of first 4 rows of ${}^{\mathcal{D}}D_4$ is poor, a singular value of .07 is present. But the matrix made from rows 1,2,3 and 5 is well conditioned so choose $\mathcal{B} = [1, x, y, xy]$.

Since all basis monomials are quadratic we only need to calculate $D_3^{\mathcal{B}}$.

Proceed to $\mathbf{L}(8, \hat{\mathbf{o}})$ which is a 72×45 matrix. The four smallest singular values are approximately $0, .2 * 10^{-7}, .0002, .003$ which are good enough for our purposes. The basis change matrix from rows 1,2,3,5 is still well conditioned. Also the change in $D_3^{\mathcal{B}}$ from $k = 7$ to $k = 8$

is on the order of .004 which is also small. We calculate the multiplication matrices

$$M_x = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -.26936 & .27679 & -.18437 \\ 0 & 0 & 0 & 1 \\ 0 & .01258 & .00943 & -.27292 \end{bmatrix} \quad M_y = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & .038766 & .04113 & .00786 \\ 0 & .01034 & .01084 & .03183 \end{bmatrix}$$

We have $\|M_xM_y - M_yM_x\| = .0002$ and the calculated zeros are $(0,0), (.05908, .072978)$ and $(-.30072 \pm .10756i, -.00002 \mp .10753i)$ which agree with solutions of the system F with error about 10^{-4}.

Reading from M_x and M_y, using (7), we get the non-trivial equations

$$\begin{aligned} g1 =& x^2 + .18437xy - .27679y + .26936x \\ g2 =& x^2y + .27292xy - .0094307y + .01258x \\ g3 =& y^2 - .00786xy - .04113y - .038766x \\ g4 =& xy^2 - .03183xy - .01084y + .01034x \end{aligned}$$

However, using a least square calculation we can get the approximations

$$\begin{aligned} g2 =& (-.00713 - .99852y)g1 + (.27638 - .18411x)g3 \\ g4 =& (.03871 + .00785y)g1 + (.00217 + .99855x)g3 \end{aligned}$$

with error again approximately 10^{-4}. Thus we can conclude that $g1, g3$ generate the ideal of the numerical local ring. In particular the full solution to the system $[g1, g2]$ consists of the 4 small solutions of the system F identified above.

We attempt this calculation again. This time we note the gap between singular values .02 and .20 of $\mathbf{L}(4, \hat{\mathbf{o}})$ and choose $m = 3$. While we again get small singular values and small change in $D_4^{\mathcal{B}}$, $\mathcal{B} = [1, x, y]$, the commutator is $\|M_xM_y - M_yM_x\| = .024$ which is larger than expected. Applying the method of [2] one gets 3 zeros, two of which are the real zeros of F but the third is not near a zero of the system F. Therefore the method is not successful for $m = 3$. In this case the reason is clear, M_x is real and has two real eigenvalues which are the x-coordinates of the real zeros of F. The third eigenvalue must also be real but there is no third real zero of F near $\hat{\mathbf{o}}$. A similar example of a false zero was given in [2, §7], this false zero has distance only 0.30 from the center while the one in [2] had distance 0.95 from the center.

It should be noted that replacing real $\hat{\mathbf{o}}$ with a random complex point nearby allows $m = 3$ to be used successfully to get the two real zeros of F and the closest complex zero. ∎

6 Conclusion

My goal was given a holomorphic system $F = [f_1, \ldots, f_n]$ in s complex variables and a point $\hat{\mathbf{x}} \in \mathbb{C}^s$, not necessarily a zero of the system, to construct $\mathcal{H}(\mathcal{U} \cap V_F)$ for some small $\mathcal{U}$ containing $\hat{\mathbf{x}}$ as the coordinate ring of a system of polynomial equations. Such a ring could be defined

by a closed linear space of differential functionals on $\mathbb{C}[\mathbf{x}]$ which vanish on the ideal of the ring. While an approxi-nullspace of $\mathbf{L}(k, \hat{\mathbf{x}})$ is a good candidate for such a space of differential functionals, in practice I don't obtain a close enough approximation to directly obtain the desired ring. Applying the NLR method I can find zeros of the system near $\hat{\mathbf{x}}$ and work from these as in Example 2 to find a candidate for $\mathcal{H}(\mathcal{U} \cap V_F)$.

Unfortunately the method sometimes gives false zeros and possibly misses some wanted zeros. So at present I am unable to give even an estimate of the open set $\mathcal{U}$ in $\mathcal{H}(\mathcal{U} \cap V_F)$. Additional study of this issue is required. In the meantime it is my hope that the examples in [2] and this report at least illustrate that this method may be useful as one tool that can be used in an attempt to understand the local zero structure of a difficult system of equations.

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