Numerical Algebraic Geometry via Macaulay's Perspective

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Motivation

[A solution] might be regarded as in some measure complete if it were admitted that [the] problem is finished with when its solution has been reduced to a finite number of feasible operations. If however the operations are too numerous or too involved to be carried out in practice the solution is only a theoretical one; and its importance then lies not in itself, but in the theorems with which it is associated and to which it leads. **Such a theoretical solution must be regarded as preliminary and not the final stage of the consideration of the problem.**

F.S. Macaulay, The Algebraic Theory of Modular Systems, 1916

We will be using techniques developed or mentioned in Macaulay's 1916 book such as H-bases, Macaulay (dialytic) arrays Sylvester (resultant) arrays and duals (inverse functions and arrays).

H-bases

An *H-basis* was defined in 1916 by Macaulay as follows:

> *The distinctive property of an H-basis* $(F_1, F_2, \ldots, F_k)$ *of* M *is than any member* F *of* M *can be put in the form* $A_1F_1 + A_2F_2 + \cdots + A_kF_k$ *where* A_iF_i $(i = 1, 2, \ldots, k)$ *is not of greater degree than* F. *Every module [ideal] has an H-basis, which may necessarily consist of more members than would suffice for a basis in general*

Note that a homogeneous basis for a homogeneous ideal is an H-basis, the original motivation for the name *H-basis*

We will see below that H-bases are associated with maximal rank Sylvester Matrices.

H-bases appear to be an appropriate alternative to Gröbner bases in the numerical case.

Local Dual Functionals

A *Local dual functional* is a $\mathbb{C}$-linear map

$$\mathbb{C}[[\mathbf{x}]]/\mathbb{C}[[\mathbf{x}]]\mathcal{I}\Big|_{\hat{\mathbf{x}}} \longrightarrow \mathbb{C}$$

Here $\mathbb{C}[[\mathbf{x}]]/\mathbb{C}[[\mathbf{x}]]\mathcal{I}\Big|_{\hat{\mathbf{x}}}$ denotes the local ring at point $\hat{\mathbf{x}}$.
A typical such functional is given by a **finite** sum

$$\sum_{|\mathbf{j}|<n} \beta_{\mathbf{k}} \partial_{\mathbf{x}\mathbf{j}}[\hat{\mathbf{x}}], \text{ where } \partial_{\mathbf{x}\mathbf{j}}[\hat{\mathbf{x}}] \equiv \frac{1}{j_1! \cdots j_s!} \left. \frac{\partial^{j_1+\cdots+j_s}}{\partial x_1^{j_1} \cdots \partial x_s^{j_s}} \right|_{\hat{\mathbf{x}}}$$

These have been studied by Max Noether, Lasker, Macaulay, Gröbner, Mora, Möller, Mourrain, Stetter, L.Zhi et. al., and [DLZ] among others.

Global dual functionals

A *Global dual functional*, in my terminology, is a $\mathbb{C}$-linear map

$$\mathbb{C}[\mathbf{x}]/\mathcal{I} \longrightarrow \mathbb{C}$$

Let $\mathcal{G}(\mathcal{A})$ denote the $\mathbb{C}$-vector space of global duals of $\mathcal{A}$.

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Example: Let $\mathcal{A} = \mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \ldots, x_s]$. Then $\mathcal{G}(\mathcal{A})$ can be identified with the $\mathbb{C}$-vector space $\mathbb{C}[[X_1, \cdots, X_s]]$ of formal power series where, for $\mathbf{X}^{\mathbf{k}} = X_1^{k_1} \cdots X_s^{k_s}$ and $\mathbf{x}^{\mathbf{j}} = x_1^{j_1} \ldots x_s^{j_s}$,

$$\mathbf{X}^{\mathbf{k}}(\mathbf{x}^{\mathbf{j}}) = \begin{cases} 1 & \text{if} \quad \mathbf{j} = \mathbf{k}, \\ 0 & \text{if} \quad \mathbf{j} \neq \mathbf{k}. \end{cases}$$

More generally if $\mathcal{A} = \mathbb{C}[\mathbf{x}]/\mathcal{I}$ then $\mathcal{G}(\mathcal{A})$ is the subspace of $\mathcal{G}(\mathbb{C}[\mathbf{x}])$ given by $\mathcal{G}(\mathcal{A}) = \{d \mid d(f) = 0 \text{ for all } f \in \mathcal{I}\}$

Difference between local and global duals

- **Global duals** are based at the origin regardless of whether the origin is in $V(\mathcal{I})$. Global duals are series and will be represented by Macaulay matrices. Rings of analytic functions do **not** have global duals.

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Note: If $\mathcal{A}$ is an affine ring with homogeneous basis then the Macaulay and Sylvester arrays are row equivalent. In this special case the local dual at the **origin** agrees with the global dual. One can use this fact to find the global dual of a homogeneous variety.

The Functorial Property of global duals

Suppose algebraic sets $\mathcal{X} \subseteq \mathbb{A}^s$, $\mathcal{I} = I(\mathcal{X})$ and $\mathcal{Y} \subseteq \mathbb{A}^r$, $\mathcal{J} = I(\mathcal{Y})$ and

$$\phi : \mathbb{A}^s \longrightarrow \mathbb{A}^r$$

is an affine map $\phi = [f_1, \ldots, f_r]$ where $f_i : \mathbb{C}[x_1, \ldots, x_s] \to \mathbb{C}$ are polynomial functions such that $\phi(\mathcal{X}) \subseteq \mathcal{Y}$. This gives a ring map

$$\phi^* : \mathbb{C}[y_1, \ldots, y_r]/\mathcal{J} \longrightarrow \mathbb{C}[x_1, \ldots, x_s]/\mathcal{I}$$

given by $\phi^*(g) = g(f_1, \ldots, f_r)$. Then we get a linear map

$$\phi_* : \mathcal{G}(\mathbb{C}[\mathbf{x}]/\mathcal{I}) \longrightarrow \mathcal{G}(\mathbb{C}[\mathbf{y}]/\mathcal{J})$$

defined by $\phi_*(d) = d \circ \phi^*$.

It is seen that we have a covariant functor from algebraic sets $\mathcal{X}$ and affine maps to $\mathbb{C}$ vector spaces and linear maps.

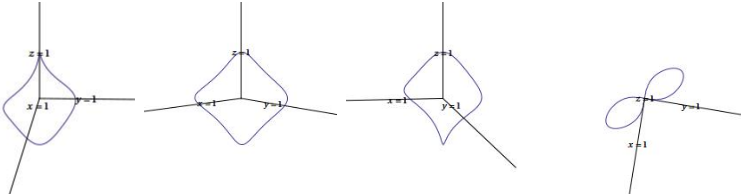
Key Algorithms

1. **Basis + point $\to$ Local Dual** There are many versions of this standard algorithm, Macaulay's and our Macaulay matrix version [DLZ] as well as *closedness* methods by Mourrain, Zeng, Zhi et. al., Hao-Sommese-Zeng etc.
2. **Local duals $\to$ Global Dual** aka *local-global method*. Calculates global dual using local dual structure at several points on a component. [D1, D2]
3. **H-basis $\to$ Global dual**
4. **Global dual of $\mathcal{X}$ $\to$ Global dual of $\overline{\phi(\mathcal{X})}$** (Functorial Property)
5. **Global Dual $\to$ Minimal H-basis** See [D1] etc.

Some more details on my versions later in this talk.

Applications: Extrinsic vs. Intrinsic representation

Extrinsic	Intrinsic	method
Implicit description of component of algebraic set	Witness points defining component	local-global $\Leftarrow$
Implicit representation of curve or surface	parametric representation	functorality $\Leftrightarrow$
Implict representation of curve, possibly not complete intersection	plot of curve	functorality $\Rightarrow$



Simple Example of Parameterizing Curve

Consider the Bow curve $f = x^4 - x^2y + y^3$.

Using MATHEMATICA

$\mathbf{fh} = \boldsymbol{x^4 - x^2yz + y^3z}$

$\boldsymbol{A} = \mathbf{\{\{1,0,0\},\{0,0,1\},\{0,1,0\}\};}$

(* put singular point at $(0,1,0)$ in $\mathbb{P}^2$ *)

$\mathbf{G = HB2GD[\{fh\},6,\{\boldsymbol{x,y,z}\}];}$

$\mathbf{H = GMap[G,\boldsymbol{A}.\{\boldsymbol{x,y,z}\},6,\{\boldsymbol{x,y,z}\},\{\boldsymbol{x,y,z}\}];}$

$\boldsymbol{B} = \mathbf{MBasis[H,4,\{\boldsymbol{x,y,z}\},1.*\text{^}\text{-}12];}$

$\boldsymbol{g} = \boldsymbol{B}\mathbf{[[1]]/.\{\boldsymbol{z} \to 1, \boldsymbol{x} \to \boldsymbol{t}\}}$

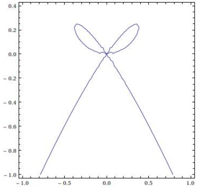
$t^4 + y - t^2y$ (* so $y = \frac{t^4}{-1+t^2}$ *)

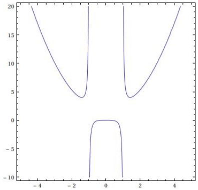
$\mathbf{gf = (\boldsymbol{y}/.Solve[\boldsymbol{g} == 0,\{\boldsymbol{y}\}])[[1]];}$

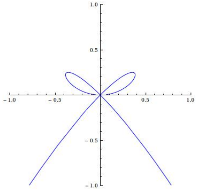
$\boldsymbol{V} = \mathbf{Inverse[\boldsymbol{A}].\{\boldsymbol{t},gf,1\};}$

$\boldsymbol{v} = \mathbf{Take[\boldsymbol{V}/\boldsymbol{V}[[3]],2]}$

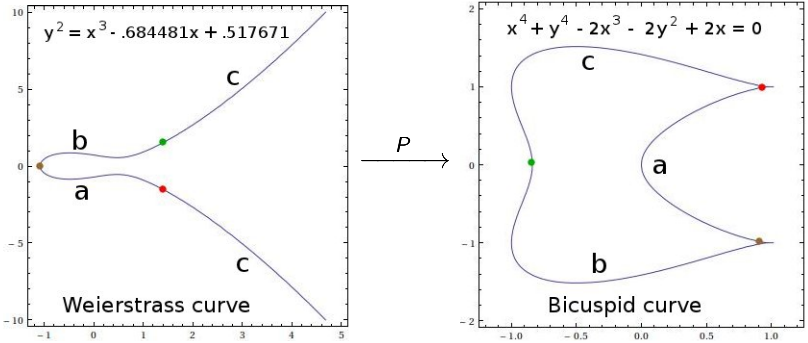
$\left\{\frac{-1+t^2}{t^3}, \frac{-1+t^2}{t^4}\right\}$ (* gives parameterization *)







Complicated Example of parameterizing curve



Here $P = \left[\dfrac{A(x,y)}{D(x,y)}, \dfrac{B(x,y)}{D(x,y)}\right]$ where, rounded,

$$A(x,y) = 0.00409784 - 2.70587x^2 + 0.830515x + 0.587122xy + 0.109105y + 0.114422y^2$$

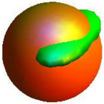
$$B(x,y) = 0.778027 + 0.585722x^2 - 2.67799x + 0.99896xy + 0.477164y - 0.408743y^2$$

$$D(x,y) = -1.80409 + 1.70004x - 0.679318x^2 + 1.84889y + 0.706692xy - 0.270131y^2$$

Comparision of Graphing Techniques

Consider the intersection of the sphere and ellipsoid

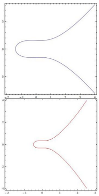
$$-16 + x^2 + y^2 + z^2 = 0$$
$$14.25 - 3x + x^2 + y^2/4 - 16z + 4z^2 = 0$$



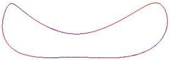
Using the functorial method of graphing this intersection we get

using two overlapping parameterizations

$$y^2 = 1.005 - 0.1453x + x^3$$
$$y^2 = 0.1127 - 0.0339x + x^3$$



$$\xrightarrow{P_1}$$
$$\xrightarrow{P_2}$$



Macaulay and Sylvester Arrays of Polynomials and Ideals

Let $f = x - y + 3y^3,\ g = z + 2x^2 - 3y^2$ in $\mathbb{C}[x, y, z],\quad F = [f, g, yg]$

The *Macaulay Array* [DLZ] of F of order 2 at the origin is

	1	x	y	z	x^2	xy	xz	y^2	yz	z^2
f	0	1	-1	0	0	0	0	0	0	0
g	0	0	0	1	2	0	0	-3	0	0
yg	0	0	0	0	0	0	0	0	1	0

Note that the rows for f, yg are truncated. If we also included rows for xf, xg, yf, zf, zg we would call this the *Macaulay array of the basis* $\langle f, g, \rangle$.

Macaulay and Sylvester Arrays Continued

- If only rows which correspond to polynomials in F of order n, i.e. not truncated, are included we call the array a *Sylvester Array of order n.* If all multiples of these rows by monomials which still have terms not exceeding total degree n are included we have the *Sylvester Array of the basis F*.
- If we include additional rows so that every polynomial of total degree n or less in the ideal generated by the listed polynomials corresponds to a vector in the rowspace then we would call this the *Sylvester Matrix of the ideal.* In the example of the previous page one would include the row corresponding to the polynomial $f + yg = x - y + z + 2x^2$.
- If the Sylvester matrix of a basis F of $\mathcal{I}$ is a Sylvester matrix of the ideal generated by the elements of F then F is an *H-basis*. In other words list F is an H-basis if the Sylvester matrix of basis F of each order has maximal rank among all Sylvester matrices of bases from $\mathcal{I}$.

Global and Local dual spaces as arrays

Local duals can be put in Sylvester type arrays, global in Macaulay type. We view the dual functionals as columns.

Consider the ideal $\langle f \rangle \subseteq \mathbb{C}[x, y]$ given by $f = x + 2y + x^2 + 3xy + y^2$. The local duals are at point $(0, 0)$, indices on right.

Local duals order 2

1	0	0	∂_1
0	-2	1	∂_x
0	1	0	∂_y
0	0	4	∂_{x^2}
0	0	-2	∂_{xy}
0	0	1	∂_{y^2}

Global duals order 2

1	0	0	0	0	1
0	-2	1	1	0	X
0	1	0	0	0	Y
0	0	4	-4	-3	X^2
0	0	-2	1	1	XY
0	0	1	0	0	Y^2

Note the last two columns of the global duals are truncated.

Local and Global Dual Principle

For large enough order

$$\begin{bmatrix} \text{Sylvester Matrix} \\ \text{of ideal } \mathcal{I} \end{bmatrix} \begin{bmatrix} \text{Macaulay} \\ \text{array of} \\ \text{global} \\ \text{duals of} \\ \mathbb{C}[\mathbf{x}]/\mathcal{I} \end{bmatrix} = 0$$

$$\begin{bmatrix} \text{Macaulay array} \\ \text{of ideal } \mathcal{I} \end{bmatrix} \begin{bmatrix} \text{Sylvester} \\ \text{matrix of} \\ \text{local} \\ \text{duals of} \\ \mathbb{C}[\mathbf{x}]/\mathcal{I} \end{bmatrix} = 0$$

In both cases the columns of the right hand matrix span the nullspace of the left hand matrix and the rows of the left hand matrix span the left nullspace of the right hand matrix.

The functorial transformation $\mathcal{G}_n(\phi)$

Assume that $\phi = [f_1, \ldots, f_r] : \mathbb{A}^s \to \mathbb{A}^r$ is a polynomial map which takes the origin to the origin. If necessary do a linear translation of variables or homogenize. Let $\mathcal{X} = V(\mathcal{I})$ be an algebraic set in $\mathbb{A}^s$ and set $\mathcal{J} = {\phi^*}^{-1}(\mathcal{I})$ so that $\mathcal{Y} = V(\mathcal{J}) = \overline{\phi(\mathcal{X})}$.

Construct $\mathcal{G}_n(\phi)$ as follows:

- For each monomial $\mathbf{y}^{\mathbf{k}}$ of total degree n or less substitute $y_i = f_i$ to get $g_{\mathbf{k}} = f_1^{k_1} \cdots f_r^{k_r} \in \mathbb{C}[x_1, \ldots, x_s]$.
- Set $\mathcal{G}_n(\phi)$ to be the Macaulay matrix of the polynomial list $[\{g_{\mathbf{k}}\}]$

Theorem: Using large enough n, with high probability

$$\mathcal{G}_n(\mathbb{C}[\mathbf{y}]/\mathcal{J}) = \mathcal{G}_n(\phi)\mathcal{G}_n(\mathbb{C}[\mathbf{x}]/\mathcal{I})$$

Local to Global

For $\mathbf{i} = [i_1, \ldots, i_s], \mathbf{j} = [j_1, \ldots, j_s]$, $\mathbf{i} \geq \mathbf{j}$ means $i_\alpha \geq j_\alpha$ for all $1 \leq \alpha \leq s$. Then as functionals on $\mathbb{C}[\mathbf{x}]$

$$\partial_{\mathbf{x}^{\mathbf{j}}}[\hat{\mathbf{x}}] = \sum_{\mathbf{i} \geq \mathbf{j}} \binom{i_1}{j_1} \hat{x}_1^{i_1 - j_1} \cdots \binom{i_s}{j_s} \hat{x}_s^{i_s - j_s} \mathbf{X}^{\mathbf{i}}$$

where $\hat{\mathbf{x}} = (\hat{x}_1, \ldots, \hat{x}_s)$. The left hand side is a local functional and the right a global functional. From a matrix point of view we have for fixed n

$$\begin{bmatrix} \text{Macaulay Matrix} \\ \text{of order } n \\ \text{global duals from } \hat{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} \text{Change of Center} \\ \text{matrix} \\ \text{of order } n \end{bmatrix} \begin{bmatrix} \text{Sylvester Matrix} \\ \text{of order } n \\ \text{local duals at } \hat{\mathbf{x}} \end{bmatrix}$$

Local to Global Theorem

Given an ideal $\mathcal{I}$ of $\mathbb{C}[x_1, \ldots, x_s]$, $n > 0$ and points $\hat{\mathbf{p}}_i,\ i = 1, \ldots, k$ of $V(\mathcal{I})$ concatenate the Macaulay matrices of order n global duals. Write $\mathcal{D}_n(\{\hat{\mathbf{p}}_1, \ldots, \hat{\mathbf{p}}_k\})$ for this matrix.

Local-Global Theorem, matrix form: *For given $n > 0$ there exist finitely many points, $\hat{\mathbf{p}}_1, \ldots, \hat{\mathbf{p}}_k$, of $V(\mathcal{I})$ so that the Sylvester matrix of the ideal $\mathcal{I}$ of order n is the left nullspace of $\mathcal{D}_n(\{\hat{\mathbf{p}}_1, \ldots, \hat{\mathbf{p}}_k\})$.*

Corollary *An H-Basis for $\mathcal{I}$ can be obtained from finitely many local duals at finitely many points of $V(\mathcal{I})$.*

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It remains an open question as to how many and what points are needed. It is clear that it is necessary to have at least one point from each component of $V(\mathcal{I})$. In principle, for large n, that may be enough. In practice more points may be needed and the number may be dependent on implementation issues as well as algebraic-geometric factors.

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