Dimension Exercises

Simple questions:

i. Assume \( \{\vec{u}, \vec{v}, \vec{w}\} \) spans a vector space \( V \). What can you say about \( \dim(V) \)?
Answer: \( \dim(V) \leq 3 \) because spanning set can be shrunk to a basis.

ii. Assume \( V \) is a vector space and \( \vec{u}, \vec{v} \) are LI vectors in \( V \). What can you say about \( \dim(V) \)?
Answer: \( \dim(V) \geq 2 \) because \( \{\vec{u}, \vec{v}\} \) is LI so it can be expanded to a basis.

iii. Assume \( \{\vec{u}, \vec{v}, \vec{w}\} \) spans a vector space \( V \) and \( 2 \cdot \vec{v} - 3 \vec{w} = \vec{0} \). What can you say about \( \dim(V) \)?
Answer: \( \dim(V) < 3 \) since \( 0 \cdot \vec{u} + 2 \cdot \vec{v} - 3 \vec{w} = \vec{0} \) implies \( \{\vec{u}, \vec{v}, \vec{w}\} \) is LD, can not be a basis but can be shrunk to a basis.

iv. Let \( V \) be a VS. If \( \vec{u}, \vec{v} \) are LI vectors in \( V \) but do not span \( V \), what can you say about \( \dim(V) \)?
Answer: \( \dim(V) > 2 \) because \( \vec{u}, \vec{v} \) are LI, not a basis, and can be expanded to a basis.

v. Let \( \{\vec{u}, \vec{v}, \vec{w}\} \) be LI in a vector space \( V \). If every vector in \( V \) is a LC of \( \vec{u}, \vec{v}, \vec{w} \), what can you say about \( \dim(V) \)?
Answer: \( \dim(V) = 3 \) because \( \{\vec{u}, \vec{v}, \vec{w}\} \) is LI and a spanning set, thus a basis.

vi. Let \( V \) be a VS of dimension 5 and \( W \) is a subspace of \( V \). What can you say about the \( \dim(W) \)?
Answer: \( \dim(W) \leq 5 \) since a subspace can't have higher dimension.

vii. Let \( V \) be a VS containing two vectors \( \vec{v} \neq \vec{0}, \vec{w} \neq \vec{0} \). What can you say about \( \dim(V) \)?
Answer: \( \dim(V) \geq 1 \), because \( \vec{v} \neq \vec{0} \) implies \( \{\vec{v}\} \) is LI and can be expanded to a basis. The other vector \( \vec{w} \neq \vec{0} \) can also be expanded to a basis, still \( \dim(V) \geq 1 \).

viii. A vector space \( V \) contains only a single vector. What can you say about \( \dim(V) \)?
Answer: The vector space containing only a single vector is the vector space \( V = \{\vec{0}\} \), so \( \dim(V) = 0 \).

Proof problems.

1. Assume \( \{\vec{u}, \vec{v}, \vec{w}\} \) spans a vector space \( V \) and there exist \( a, b \) such that \( \vec{u} + b \cdot \vec{v} + c \cdot \vec{w} = \vec{0} \).
Prove \( \dim(V) \leq 2 \).
Proof. \( \therefore \{\vec{u}, \vec{v}, \vec{w}\} \) spans \( V \)
\( \therefore \dim(V) \leq 3 \)
\( \therefore 1 \cdot \vec{u} + b \cdot \vec{v} + c \cdot \vec{w} = \vec{0} \) and 1, a, b are nontrivial
\( \therefore \{\vec{u}, \vec{v}, \vec{w}\} \) is not a basis.
\( \therefore \dim(V) \leq 2 \). QED.

2. Assume \( \{\vec{u}, \vec{v}, \vec{w}\} \) spans a vector space \( V \) and \( \dim(V) = 2 \). Prove \( \vec{u}, \vec{v}, \vec{w} \) are LD.
Proof. We shall prove the conclusion by contradiction.
Assume \( \vec{u}, \vec{v}, \vec{w} \) are LI. Then they can be expanded to a basis so \( \dim(V) \geq 3 \). This is a contradiction to \( \dim(V) = 2 \). \( \therefore \vec{u}, \vec{v}, \vec{w} \) are LD. QED.
3. If \( \{ \vec{u}, \vec{v} \} \) is a LI subset of a vector space \( V \), prove \( \dim(V) \geq 2 \).
Proof. \( \because \{ \vec{u}, \vec{v} \} \) is LI in \( V \) and thus can be expanded to a basis containing at least 2 vectors. \( \therefore \) \( \dim(V) \geq 2 \).

4. Knowing \( \{ 1 + x + x^2, 1 + x, x + x^2 \} \) is LI, prove it is a basis for \( P_2 \) using a dimension argument.
Proof. (Proof by contradiction) Assume \( \{ 1 + x + x^2, 1 + x, x + x^2 \} \) is not a basis for \( P_2 \).
\( \therefore \{ 1 + x + x^2, 1 + x, x + x^2 \} \) is LI, not a basis but can be expanded to a basis.
\( \therefore \dim(P_2) \geq 3 \), which is a contradiction to \( \dim(P_2) = 3 \).
\( \therefore \{ 1 + x + x^2, 1 + x, x + x^2 \} \) is a basis for \( P_2 \). QED.

5. Knowing \( \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \) is a spanning set for \( M_{2 \times 2} \), prove it is LI using a dimension argument.
Proof. (proof by contradiction) Assume the set is LD and is thus not a basis.
\( \therefore \) The set spans \( M_{2 \times 2} \).
\( \therefore \) It can be shrunk to a basis
\( \therefore \dim(M_{2 \times 2}) < 4 \) which is a contradiction to \( \dim(M_{2 \times 2}) = 4 \).
\( \therefore \) It is LI. QED.

6. Knowing \( \begin{bmatrix} 2 \\ 0 \\ 3 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} \) is not a spanning set for \( R^3 \), prove it is LD using a dimension argument.
Proof. (proof by contradiction) Assume the set is LI and thus can be expanded to a basis.
\( \therefore \dim(R^3) = 3 \) and thus no more vectors can be added into the set to expanded it to basis of 4 or more vectors.
\( \therefore \) The set is a basis already, which is a contradiction to it not being a spanning set.
\( \therefore \) It is LD. QED.

7. Prove \( \{ 1 + x, x + x^2, 1 - x^2, 1 + 2 \cdot x + x^2 \} \) is LD using a dimension argument.
Proof. (proof by contradiction) Assume the set is LI and thus can be expanded to a basis.
\( \therefore \dim(P_2) \geq 4 \) that is a contradiction to \( \dim(P_2) = 3 \).
\( \therefore \) It is LD. QED.

8. Assume \( \{ \vec{u}, \vec{v}, \vec{w} \} \) spans a vector space \( V \) and \( \vec{x}, \vec{y}, \vec{z} \in V \) are LI. Prove \( \vec{u}, \vec{v}, \vec{w} \) are LI.
Proof. (proof by contradiction) Assume \( \vec{u}, \vec{v}, \vec{w} \) are LD and thus not a basis.
\( \therefore \{ \vec{u}, \vec{v}, \vec{w} \} \) spans \( V \) and thus can be shrunk to a basis
\( \therefore \dim(V) < 3 \).
\( \therefore \vec{x}, \vec{y}, \vec{z} \in V \) are LI and thus can be expanded to a basis.
\( \therefore \dim(V) \geq 3 \), contradicting to \( \dim(V) < 3 \).
\( \therefore \vec{u}, \vec{v}, \vec{w} \) are LI. QED.
9. Assume \( \{ \vec{u}, \vec{v}, \vec{w}, \vec{z} \} \) spans a vector space \( V \) with dimension 4. Prove \( \{ \vec{u}, \vec{v}, \vec{w}, \vec{z} \} \) is a basis for \( V \).

**Proof.** (proof by contradiction) Assume the set is not a basis for \( V \).

\[ \dim(V) < 4 \] since the set is not a basis but can be shrunk to a basis, contradicting to \( \dim(V) = 4 \).

\( \therefore \) The set is a basis for \( V \).

10. Assume \( \{ \vec{u}, \vec{v}, \vec{w}, \vec{p} \} \) spans a vector space \( V \). If \( V \) has another spanning set \( \{ \vec{x}, \vec{y}, \vec{z} \} \), prove \( \vec{u}, \vec{v}, \vec{w}, \vec{p} \) are LD.

**Proof.** (proof by contradiction) Assume \( \vec{u}, \vec{v}, \vec{w}, \vec{p} \) are LI. Then they can be expanded to a basis.

\[ \dim(V) \geq 4 \]

\[ \therefore \{ \vec{x}, \vec{y}, \vec{z} \} \] is a spanning set for \( V \) and can be shrunk to a basis

\[ \dim(V) \leq 3, \] contradicting to \( \dim(V) \geq 4 \).

\( \therefore \) \( \vec{u}, \vec{v}, \vec{w}, \vec{p} \) are LD. **QED.**