Basis exercise

1. Assume \{\vec{u}, \vec{v}, \vec{w}\} spans a vector space \(V\) and none of \(\vec{u}, \vec{v}, \vec{w}\) is a LC of the other two vectors. Prove \{\vec{u}, \vec{v}, \vec{w}\} is a basis for a vector space \(V\). Prove \{\vec{u}, \vec{v}, \vec{w}\} is a basis for \(V\).

2. Assume \{\vec{u}, \vec{v}, \vec{w}\} spans a vector space \(V\). If the only \(a, b, c\) that make \(a \cdot \vec{u} + b \cdot \vec{v} + c \cdot \vec{w} = \vec{0}\) are \(a = b = c = 0\), prove \{\vec{u}, \vec{v}, \vec{w}\} is a basis for \(V\).

3. If \{\vec{u}, \vec{v}, \vec{w}\} is a basis for vector space \(V\) and \(a \cdot \vec{u} + 3 \cdot \vec{v} = b \cdot \vec{v} + c \cdot \vec{w}\), prove \(a = 0, \ b = 3, \ c = 0\)

4. If \{\vec{u}, \vec{v}, \vec{w}\} is a basis for a vector space \(V\) and \{5 \cdot \vec{u}, 3 \cdot \vec{v}\} spans a vector space \(W\), prove \{5 \cdot \vec{u}, 3 \cdot \vec{v}\} is a basis for \(W\)

5. If \{\vec{u}, \vec{v}\} is a basis for a vector space \(V\), prove \{3 \cdot \vec{u}, \frac{1}{2} \cdot \vec{v}\} is also a basis for \(V\).

6. Assume none of \(\vec{u}, \vec{v}, \vec{w}\) are zero vectors. If \{\vec{u}, \vec{v}\} spans a vector space \(V\) and \{\vec{w}\} also spans \(V\), prove \(\vec{u}, \vec{v}\) are LD.

7. Can you prove the statement in Problem 6 without the assumption "none of \(\vec{u}, \vec{v}, \vec{w}\) are zero vectors"?

8. If \{\vec{u}, \vec{v}, \vec{w}\} spans a vector space \(V\) and \(\vec{u} = 3 \cdot \vec{v}\), prove \{\vec{u}, \vec{v}, \vec{w}\} is not a basis.

9. Assume \{\vec{u}, \vec{v}\} spans a vector space \(V\) and \{\vec{u} + \vec{v}, \vec{v}\} is LI. Prove \{\vec{u}, \vec{v}\} is a basis for \(V\).

10. Assume \{\vec{u}, \vec{v}\} spans a vector space \(V\) and \{\vec{u} + \vec{v}, \vec{v}\} is LD. Prove \{\vec{u}, \vec{v}\} is not a basis for \(V\).

11. If \{\vec{u}, \vec{v}, \vec{w}\} spans a vector space \(V\) and \(\vec{w}\) is a LC of \(\vec{u}, \vec{v}\), prove \(\vec{u}, \vec{v}\) also span \(V\)

12. If \{\vec{u}, \vec{v}, \vec{w}\} spans a vector space \(V\) and \(\vec{w} \neq \vec{0}\), prove \{\vec{w}\} is a basis for \(V\).

13. If \{\vec{u}, \vec{v}, \vec{w}\} spans a vector space \(V\) and \(\vec{w} = \vec{0}\), prove \{\vec{u}, \vec{v}\} also spans \(V\).

14. If \{\vec{u}, \vec{v}, \vec{w}\} is a basis for a vector space \(V\) and \((a + b) \cdot \vec{u} + (b + c) \cdot \vec{v} + c \cdot \vec{w} = \vec{0}\), prove \(a = b = c = 0\).

15. If \{\vec{u}, \vec{v}, \vec{w}\} is a basis for a vector space \(V\), prove \(\vec{w} \neq \vec{0}\).

16. If \{\vec{u}, \vec{v}, \vec{w}\} spans a vector space \(V\) and \(2 \cdot \vec{u} + 3 \cdot \vec{v} = b \cdot \vec{v} + c \cdot \vec{w}\), prove \{\vec{u}, \vec{v}, \vec{w}\} not a basis.