Linear dependence/independence exercises

1. Let \( \vec{u}, \vec{v}, \vec{w}, \vec{p} \) be vectors in a vector space and \( \vec{p} \) is a linear combination of \( \vec{u} \) and \( \vec{v} \). Prove: \( \vec{p} \) is a linear combination of \( \vec{u}, \vec{v} \) and \( \vec{w} \).

   \[ \vec{p} = a \vec{u} + b \vec{v} \]

   Prove: \( a \vec{u} + b \vec{v} = c \vec{w} \). \( \vec{p} \) is a linear combination of \( \vec{u}, \vec{v} \) and \( \vec{w} \).

2. Prove: The zero vector is linearly dependent.

   \[ \vec{0} = a \vec{u} + b \vec{v} \]

   \[ a = b = 0 \]

3. Prove: If \( \vec{u} \neq \vec{0} \), then \( \vec{u} \) is linearly independent.

4. Prove: If \( \vec{u} \) and \( \vec{v} \) are linearly independent, also \( a \vec{u} + b \vec{v} = c \vec{u} + d \vec{v} \) then \( a = c \) and \( b = d \).

5. Prove: If \( \vec{u} \) and \( \vec{v} \) are linearly dependent and \( \vec{v} \neq \vec{0} \), then there is a scalar \( c \) such that \( \vec{u} = c \vec{v} \).

6. Prove: Vectors \( \vec{u}, \vec{v}, \vec{0} \) are linearly dependent.

7. Prove: If \( \vec{u}, \vec{v}, \vec{w} \) are linearly independent, then \( 2 \vec{u}, 3 \vec{v}, 4 \vec{w} \) are also linearly dependent.

8. Prove: If \( \vec{u}, \vec{v} \) are linearly dependent, also \( a \vec{u} + b \vec{v}, c \vec{w} \) are also linearly dependent.

9. Prove: If \( \{\vec{u}, \vec{v}\} \) and \( \{\vec{u}, \vec{v}, \vec{w}\} \) span the same vector space, then \( \vec{w} \) is a linear combination of \( \vec{u} \) and \( \vec{v} \).

10. Prove: If \( \vec{p} \) is a linear combination of \( \vec{u}, \vec{v}, \vec{w} \) and \( \vec{w} \) is a linear combination of \( \vec{u}, \vec{v}, \vec{p} \), then \( \vec{u}, \vec{v}, \vec{p} \) are linearly dependent.

11. Prove: If \( \vec{u}, \vec{v} \) are linearly dependent, then \( \vec{u}+\vec{v}, \vec{u}-\vec{v} \) are linearly dependent.

12. Prove: If \( \vec{u}, \vec{v} \) are linearly independent, then \( \vec{u}+\vec{v}, \vec{u}-\vec{v} \) are linearly independent.

13. Prove: If \( \vec{u}, \vec{v} \) are linearly dependent while \( \vec{u} \) is not a linear combination of \( \vec{v} \), then \( \vec{v} = \vec{0} \).

14. Prove: If \( \vec{u}, \vec{v}, \vec{w} \) are linearly dependent while \( \vec{w} \) is not a linear combination of \( \vec{u}, \vec{v} \), then \( \vec{u}, \vec{v} \) are linearly dependent.

15. Prove: If \( \vec{u}, \vec{v} \) are linearly independent while \( \vec{u}, \vec{v}, \vec{w} \) are linearly dependent, then \( \vec{w} \) is a linear combination of \( \vec{u}, \vec{v} \).

16. Prove: If \( \{\vec{u}, \vec{v}\} \) and \( \{\vec{w}\} \) span the same vector space, then \( \vec{u}, \vec{v} \) are linearly dependent.

17. Prove: If \( \vec{u}, \vec{v}, \vec{w} \) are all linear combinations of \( \vec{p}, \vec{q} \), then \( \vec{u}, \vec{v}, \vec{w} \) are linearly dependent.